More Examples of Solutions to Differential Equations

(1) Solve $\frac{dy}{dx} = xy^2$. when $y(0) = 1$

**Solution:**

$$\int \frac{1}{y^2} dy = \int x dx$$

so we get

$$-\frac{1}{y} = \frac{x^2}{2} + c$$

so

$$y = \frac{-1}{\frac{1}{2}x^2 + c}.$$  

Also, $y = 0$ is also a solution.

(2) (Logistic Equation) This equation describes the change in population size in which the growth rate depends on the density of the population. It satisfies the following differential equation

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$

where $r, K$ are positive constants, $r$ is the per capita growth rate at time $t = 0$ and $K$ the carrying capacity of the population.

**Solution:** If $N \neq 0$ and $N \neq K$ we have

$$\int \frac{dN}{(1 - \frac{N}{K})N} = \int r dt$$

Note that

$$\frac{1}{(1 - \frac{N}{K})N} = \frac{K}{N(K - N)}$$

using partial fractions we get

$$\frac{K}{N(K - N)} = \frac{1}{N} + \frac{1}{K - N}$$

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Integrating this, we get

$$\ln|N| - \ln|K - N| = rt + C_1$$

$$\ln\left|\frac{N}{K - N}\right| = rt + C_1 - \ln\left|\frac{K - N}{N}\right| = rt + C_1$$

$$\ln|\frac{K - N}{N}| = -rt - C_1$$

$$\left|\frac{K - N}{N}\right| = e^{-rt-C_1}$$

$$\frac{K - N}{N} = Ce^{-rt}$$

Then we get

$$N = \frac{K}{1 + Ce^{-rt}}$$

Note that $N = 0$ and $N = K$ are also solutions.

(3) (von Bertalanffy growth equation) Consider a growing organism, such as a fish, and let $L(t)$ be its length at time $t$ in cm. We can think of $t$ as the age of the fish. Let $L_\infty$ be the maximum size of the fish, its asymptotic length. Suppose the growth rate of the fish depends on the difference $L_\infty - L$ so we have

$$\frac{dL}{dt} = k(L_\infty - L)$$

where $k$ is some constant. Find the family of solutions of this then find the solution that has initial length $L(0) = 3$

**Solution:** $L = L_\infty - Ce^{-kt}$.

(4) (Allometric growth) Allometric growth is how different parts of an organism grow dependent on another. For example, a crab’s body width and the claw length. In general, The growth rates are proportional to the sizes. Let $L_1$ and $L_2$ be the size of two body parts, then

$$\frac{dL_1}{dt} = k \frac{L_1}{L_2}$$

(5) For example, consider the crab claw problem. Let $L$ be the claw length and $B$ the body length. We have

$$\frac{dL}{dt} \frac{1}{L} = 1.57 \frac{dB}{dt} \frac{1}{B}$$

**Solution:** $L = kB^{1.57}$
(6) (Solution Mixing Problems)

These problems typically have a tank with a solution coming in at a fixed rate, and the tank is evenly mixed, and leaves the tank at a fixed rate. These problems we usually suppose the inflow and outflow rates are the same. Let \( r \) be the flow rates, "b" be the solution concentration coming in, \( V \) the volume of the tank, and \( y(t) \) is the amount of substance in the tank. Then \( \frac{dy}{dt} = \) (rate in)-(rate out), or more precisely with this model,

\[
\frac{dy}{dt} = rb - \frac{ry}{V}
\]

Let’s try a problem. (Problem 45 in 7.4) A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/m. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after \( t \) minutes? (b) after 20 minutes?

**Solution:**

Let \( S(t) \) be the amount of salt in the tank.

\[
\frac{dS(t)}{dt} = 10(0) - (10) \left( \frac{S(t)}{1000} \right)
\]

solving this part way we get,

\[
\ln|S| = -\frac{1}{100} t + c
\]

Using the initial condition that \( S(0) = 15 \) kg, we get that \( c = \ln|15| \)

\[
S = e^{c}e^{-\frac{1}{100}t}
\]

so the solution is

\[
S(t) = 15e^{-\frac{1}{100}t}
\]

so after 20 minutes,

\[
S(20) = 15e^{-2/10}
\]

which is about 12.3 kg.