Technical Appendix: Productivity and the role of the global acquisition market

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1 Open Economy - Small Trade Costs - Endogenous A

This section will be identical to the main paper with the exception of allowing A to be determined endogenously. I assume that the conditions of Lemma 3 are satisfied. To summarize:

1. All active firms can export
2. Domestic and foreign acquisitions are not trivially unprofitable relative to the acquisition price.
3. $\frac{(2A-t)t}{4b} - \delta < R_a$

Subject to these conditions, I will rewrite the key equations from the paper. The profits of doing nothing are written as:

$$\pi^{N,STC}(\alpha) = \frac{A^2\alpha k}{2(b\alpha k + v)} - \frac{t(4bA\alpha k - 2b\alpha k t - vt)}{8b(b\alpha k + v)} \tag{1}$$

The profits of buying a domestic firm (B) are:

$$\pi^{B,STC}(\alpha) = \frac{A^2\alpha k}{(2b\alpha k + v)} - \frac{t(8bA\alpha k - 4b\alpha k t - vt)}{8b(2b\alpha k + v)} \tag{2}$$

Finally, the profits from buying a foreign firm (B^*) are:

$$\pi^{B^*}(\alpha) = \pi^{B^*,FT}(\alpha) = \frac{A^2\alpha k}{(2b\alpha k + v)}.$$

Also recall that the incentives to acquire a domestic and foreign firm, respectively, are written as:

$$\begin{aligned} \Delta\Pi(\alpha, A) &= \pi^{B,STC}(\alpha, A) - \pi^{N,STC}(\alpha, A) \\ &= \left(\frac{A^2\alpha v k}{2(b\alpha k + v)(2b\alpha k + v)}\right) + \underbrace{\left(-\frac{tk\alpha(4A-t)}{8(b\alpha k + v)(2b\alpha k + v)}\right)}_{<0} \end{aligned} \tag{3}$$

$$\begin{aligned}\Delta\Pi^{*}(\alpha,A) &= \pi^{B^{*}}(\alpha,A)-\pi^{N,STC}(\alpha,A)-\delta \\ &= \left(\frac{A^{2}\alpha vk}{2\,(b\alpha k+v)\,(2b\alpha k+v)}-\delta\right)+\underbrace{\frac{t(4bA\alpha k-2b\alpha kt-vt)}{8b(b\alpha k+v)}}_{>0\text{ for exporters}}\end{aligned} \tag{4}$$

Using these five functions, I can re-write relevant productivity cutoffs and the corresponding preference conditions over acquisition options. Firms are indifferent between doing nothing and selling at α_S. This cutoff is defined by the following:

$$\pi^{N,STC}(\alpha_S,A)=R_a \tag{5}$$

where,

$$For\ \alpha<\alpha_S\ ,\ S\succ N \tag{6}$$

At $\underline{\alpha}_B$ and $\overline{\alpha}_B$, firms are indifferent between doing nothing and buying domestic capital. Precisely, these cutoffs are defined by the following equations:

$$\Delta\Pi(\underline{\alpha}_B,A) = R_a \tag{7}$$

$$\Delta\Pi(\overline{\alpha}_B,A) = R_a \tag{8}$$

where,

$$For\ \alpha\in(\underline{\alpha}_B,\overline{\alpha}_B)\,,B\succ N \tag{9}$$

As was the case under the assumption of free trade, firms within a mid-range of productivity find a domestic acquisition profitable. The existence of $\underline{\alpha}_B$ and $\overline{\alpha}_B$ will be guaranteed by condition 2.

Similarly, defining $\underline{\alpha}_B^*$ and $\overline{\alpha}_B^*$ as the levels of productivity such that firms are indifferent between foreign acquisitions and doing nothing, the indifference between these two options is summarized by the following:

$$\Delta\Pi^{*}(\underline{\alpha}_B^{*},A) = R_a \tag{10}$$

$$\Delta\Pi^{*}(\overline{\alpha}_B^{*},A) = R_a \tag{11}$$

where,

$$For\ \alpha\in(\underline{\alpha}_B^{*},\overline{\alpha}_B^{*})\,,\ B^{*}\succ N \tag{12}$$

The existence of two productivity cutoffs, $\underline{\alpha}_B^*$ and $\overline{\alpha}_B^*$, will be guaranteed by conditions 2 and 3.

Finally, for $\frac{(2A-t)t}{4b}>\delta$, the indifference point between foreign acquisitions and domestic acquisitions, α_{BB^*}, is defined as follows:

$$\Delta\Pi(\alpha_{BB^{*}},A)-\Delta\Pi^{*}(\alpha_{BB^{*}},A) = 0 \tag{13}$$

$$\alpha_{BB^{*}} = \frac{v(t^{2}+8\delta b)}{4bk((2A-t)t-4\delta b)}$$

where,

$$\frac{\partial\left(\Delta\Pi(\alpha,A)-\Delta\Pi^{*}(\alpha,A)\right)}{\partial\alpha}=-\frac{vtk\,(4A-t)}{4\,(2b\alpha k+v)^{2}}<0.$$

Thus,

$$For\ \alpha>\alpha_{BB^{*}}\ ,\ B^{*}\succ B. \tag{14}$$

With the incentives for acquisitions in-hand, I now formally define the region over which these incentives are well-defined. Precisely, Lemma 1 defines the region of t and δ that must satisfy OE1-OE4 for any equilibrium A and R_a.

Lemma 1 *There exists a $\widehat{t}$ and $\widehat{\delta}$ such that, over the space $\left[0,\widehat{t}\right] \times \left[0,\widehat{\delta}\right]$, any acquisition stage equilibrium must satisfy conditions one through three..*

Proof. Conditions 1-3 are satisfied by the same conditions as in the main body of the paper. After establishing a simple result that A and R_a are always finite and greater than zero, I will show that there exists a $\widehat{t}$ and $\widehat{\delta}$ such that over the space $\left[0,\widehat{t}\right] \times \left[0,\widehat{\delta}\right]$, any equilibrium must satisfy Conditions 1-3.

The existence of R_a is straightforward. Acquisition demand is decreasing (from positive demand) starting from $R_a = 0$. Eventually, acquisition demand is 0 with R_a high enough. This is since the incentives derived in $\Delta\Pi(\alpha)$ and $\Delta\Pi^*(\alpha)$ are bounded above. The opposite is the case with acquisition supply. For $R_a = 0$, no firms sell. For R_a high enough, all firms sell. Thus, using the intermediate value theorem, there must exist a $R_a > 0$ such that the acquisition market clears. To establish the existence of A, the demand level can be derived from the following:

$$A = \frac{\theta\gamma}{\eta 2M_E\left(1-G\left(\alpha_S\left(A\right)\right)\right)+\gamma} + \frac{\eta 2M_E\left(1-G\left(\alpha_S\left(A\right)\right)\right)}{\eta 2M_E\left(1-G\left(\alpha_S\left(A\right)\right)\right)+\gamma}\overline{p} \tag{15}$$

Defining $\overline{p} = \frac{1}{(1-G(\alpha_S(A)))}\overline{p}_u$, I can rearrange (15) as,

$$\begin{aligned} A(\eta M_E\left(1-G\left(\alpha_S\right)\right)+\frac{\gamma}{2}) &= \frac{\theta\gamma}{2}+\eta M_E\overline{p}_u \\ LHS(A) &= RHS(A) \end{aligned} \tag{16}$$

Given preferences, A is bounded between 0 and θ. As $A \rightarrow 0$, this yields the following:

$$\lim_{A\rightarrow 0} A(\eta M_E\left(1-G\left(\alpha_S\right)\right)+\frac{\gamma}{2}) = 0 < \frac{\theta\gamma}{2} = \frac{\theta\gamma}{2} + \underbrace{\lim_{A\rightarrow 0}\eta M_E\overline{p}_u}_{=0}$$

As $A \rightarrow \theta$, it is straightforward to show that:

$$\begin{aligned} \lim_{A\rightarrow\theta} A(\eta M_E\left(1-G\left(\alpha_S\right)\right)+\gamma) &= \theta\eta M_E\left(1-G\left(\alpha_S\right)\right)+\frac{\theta\gamma}{2} \\ &> \frac{\theta\gamma}{2}+\eta M_E\overline{p}_u(\theta) = \frac{\theta\gamma}{2}+\lim_{A\rightarrow\theta}\eta M_E\overline{p}_u \end{aligned}$$

This result follows by showing that $\theta\left(1-G\left(\alpha_S\right)\right) > \overline{p}_u(\theta)$. This can be rearranged as $\theta > \overline{p}(\theta)$. Since θ is the maximum price that a firm can charge, the average price $\overline{p}(\theta)$ will be below this level. Thus, given a continuous distribution of productivity, and using the intermediate value theorem, RHS and LHS must cross at least once on the interior, $A \in (0,\theta)$.

Defining the equilibrium values of A and R_a as $A\left(t,\delta\right)$ and $R_a\left(t,\delta\right)$, I now formally define $\left[0,\widehat{t}\right] \times \left[0,\widehat{\delta}\right]$. First, I will define the space $\left[0,\widehat{t}\right]$ for any δ. Then, over the space $\left[0,\widehat{t}\right]$, I will define $\left[0,\widehat{\delta}\right]$.

Step 1: Fix δ. To satisfy condition 3, any region of t must satisfy $\frac{(2A(t,\delta)-t)t}{4b} < R_a\left(t,\delta\right)$. Note that $\lim_{t\longrightarrow 0}\frac{(2A(t,\delta)-t)t}{4b} = 0$. Since $R_a\left(t,\delta\right) > 0$, then there must exist a $t_1(\delta) > 0$ such that

$\frac{(2A(t,\delta)-t)t}{4b} < R_a(t,\delta)$ for $t \in [0, t_1(\delta)]$. Next, regarding domestic acquisitions, condition 2 requires that $\frac{(3-2\sqrt{2})}{8b}(2A(t,\delta)-t)^2 > R_a(t,\delta)$. It is possible that for some t, $\frac{(3-2\sqrt{2})}{8b}(2A(t,\delta)-t)^2 < R_a(t,\delta)$. However, the free trade equilibrium developed in Spearot (2008) dictates that $\frac{(3-2\sqrt{2})}{2b}A^2(0,\delta) > R_a(0,\delta)$. This is convenient, since $\underset{t\longrightarrow 0}{lim}\frac{(3-2\sqrt{2})}{8b}(2A(t,\delta)-t)^2 = \frac{(3-2\sqrt{2})}{2b}A(0,\delta)^2$. Thus, there must exist a $t_2(\delta) > 0$ such that $\frac{(3-2\sqrt{2})}{8b}(2A(t,\delta)-t)^2 > R_a(t,\delta)$ holds for $t \in [0, t_2(\delta)]$. Conditions 2 and 3 are jointly satisfied for $t \in [0, \widetilde{t}(\delta)]$, where $\widetilde{t}(\delta) = \min\{t_1(\delta), t_2(\delta)\} > 0$.

The above a paragraph pins down an upper bound $\widetilde{t}(\delta) > 0$ for each value of δ. Thus, the minimum upper bound of t can be characterized as follows:

$$\widehat{t} = \min_{\delta}\left\{\widetilde{t}(\delta)\right\} > 0$$

Before moving to step two, first note that $\underset{\alpha}{\max}\Delta\Pi^*(\alpha|t=0) = \frac{(3-2\sqrt{2})}{2b}A(0,\delta)^2 - \delta$. Since $\frac{\partial\Delta\Pi^*(\alpha)}{\partial t} > 0$ for firms that can export, this implies that $\underset{\alpha}{\max}\Delta\Pi^*(\alpha) \geq \underset{\alpha}{\max}\Delta\Pi^*(\alpha|t=0)$. Thus, to satistfy foreign-side of condition 2, it is sufficient to show that $\underset{\alpha}{\max}\Delta\Pi^*(\alpha) = \frac{(3-2\sqrt{2})}{2b}A(t,\delta)^2 - \delta > R_a(t,\delta)$ is satisfied. This is will prove to be a useful relationship to use below.

Step 2: Fix $t \in [0, \widehat{t}]$. Over this region of trade costs, $\frac{(3-2\sqrt{2})}{8b}(2A(t,\delta)-t)^2 > R_a(t,\delta)$. Since $\frac{(3-2\sqrt{2})}{2b}A(t,\delta)^2 > \frac{(3-2\sqrt{2})}{8b}(2A(t,\delta)-t)^2$, this implies that $\frac{(3-2\sqrt{2})}{2b}A(t,\delta)^2 > R_a(t,\delta)$ over this same region. Thus, there must exist a $\widetilde{\delta}(t) = \frac{(3-2\sqrt{2})}{8b}(2A(t,\delta)-t)^2 - R_a(t,\delta) > 0$ such that for $\delta \in [0, \widetilde{\delta}(t)]$, $\frac{(3-2\sqrt{2})}{2b}A(t,\delta)^2 - \widetilde{\delta}(t) > R_a(t,\delta)$. Thus, over the region Thus, the minimum upper bound of δ can be characterized as follows:

$$\widehat{\delta} = \min_{t\in[0,\widehat{t}]}\left\{\widetilde{\delta}(t)\right\} > 0$$

Thus, independent of A and R_a, there exists a subspace $\left[0, \widehat{t}\right] \times \left[0, \widehat{\delta}\right]$ such that conditions 1-3 are satisfied. This completes the proof. ■

Since we are holding A and R_a fixed, optimal acquisition choice is idenitical to the main body of the paper. The acquisition market clearing condition is also identical, where in the main body, we are taking A as given. Thus, I now examine and discuss the conditions such that A is uniquely determined , conditional on acquisition choice described in Proposition 1, and the unique R_a characterized in Proposition 2.

Defining $\Theta_N = \underline{\Theta}_N \bigcup \overline{\Theta}_N$, A is defined by:

$$A = \frac{\theta\gamma}{\eta M + \gamma} + \frac{\eta M}{\eta M + \gamma}\overline{p}(A, \Theta_S, \Theta_N, \Theta_B, \Theta_{B^*}) \qquad (17)$$

where,

$$\overline{p}(A, \Theta_S, \Theta_N, \Theta_B, \Theta_{B^*}) = \int_{\alpha\in\Theta_N}\frac{P_N^H(\alpha) + P_N^X(\alpha)}{2(1-G(\alpha_S))}dG + \int_{\alpha\in\Theta_B}\frac{P_B^H(\alpha) + P_B^X(\alpha)}{2(1-G(\alpha_S))}dG + \int_{\alpha\in\Theta_{B^*}}\frac{P^{B^*}(\alpha) + P^{B^*}(\alpha)}{2(1-G(\alpha_S))}dG \qquad (18)$$

and,

$$M = 2M_E \left(1 - G\left(\alpha_S\left(A\right)\right)\right)$$

Here, P_N^H and P_B^H are the home price of varieties for firms that acquire nothing and buy domestic capital, respectively. P_N^X and P_B^X are the export prices resulting from these same acquisition outcomes. P^{B^*} is the price of varieties for firms that engage in foreign acquisitions. These prices are written as:

$$\begin{aligned} P_N^H &= \frac{2A\left(b\alpha k + 2v\right) - vt}{4(b\alpha k + v)}, P_N^X = \frac{2A\left(b\alpha k + 2v\right) + 2b\alpha kt + vt}{4(b\alpha k + v)} \\ P_B^H &= \frac{4A\left(b\alpha k + v\right) - vt}{4(2b\alpha k + v)},\ P_B^X = \frac{4A\left(b\alpha k + v\right) + 4b\alpha kt + vt}{4(2b\alpha k + v)} \\ P^{B^*} &= \frac{A\left(b\alpha k + v\right)}{(2b\alpha k + v)} \end{aligned}$$

Different from the free trade open economy, the productivity cutoffs will not be independent of the demand level, A. This will complicate the analysis of A, where uniqueness will require assumptions over the distribution governing productivity. However, the following lemma summarizes that the existence of A is guaranteed, independent of the productivity distribution.

Lemma 2 *Given optimal acquisition choice and acquisition market clearing, there exists a value $\widehat{A}$ that satisfies (17).*

Proof. See proof for lemma 1. ■

The result in Lemma 2 is guaranteed by the intermediate value theorem and the continuous distribution of productivity.

While the existence of $\widehat{A}$ is straightforward, uniqueness is not. To guarantee that $\widehat{A}$ is unique without additional restrictions on model parameters, an assumption must be placed on the distribution governing productivity. Generally, this assumption can be written as:

D1 For $\alpha \in [\alpha_S, \infty)$, $g\left(\alpha\right)$ is "small".

Functionally, the assumption in D1 limits the size of the indirect effects when differentiating (17) with respect to A. Assumption D1 will be satisfied by distributions of productivity that are relatively disperse. For example, an exponential distribution, $g\left(\alpha\right) = \lambda e^{-\lambda\alpha}$, with small enough λ will satisfy this assumption. Similarly, a Pareto distribution, $g\left(\alpha\right) = h\left(\frac{\widetilde{\alpha}^h}{\alpha^{h+1}}\right)$, with small enough h will also satisfy this assumption.[1] Assuming that D1 is satisfied, uniqueness is proven in the following lemma.

Lemma 3 *Given optimal acquisition choice, acquisition market clearing condition, and assuming D1, there exists a unique $\widehat{A}$ that satisfies (17).*

Proof. Existence of A is proven in lemma 3. To prove uniqueness, we can rewrite the equation pinning down A as:

$$\begin{aligned} A((1 - G\left(\alpha_S\right)) + \frac{\gamma}{2\eta M_E}) &= \frac{\theta\gamma}{2\eta M_E} + \overline{p}_u \\ LHS(A) &= RHS(A) \end{aligned} \tag{19}$$

[1] The model can be adapted to allow for a non-binding lower bound of productivity, $\widetilde{\alpha}$.

Since by Lemma 1, $LHS(A) < RHS(A)$ at $A = 0$ and $LHS(A) > RHS(A)$ at $A = \theta$, a sufficient condition for uniqueness is that $\frac{\partial LHS(A)}{\partial A} > \frac{\partial RHS(A)}{\partial A}$. I now formalize how D1 ensures this inequality, showing that $\frac{\partial LHS(A)}{\partial A} > \frac{\partial RHS(A)}{\partial A}$ for $t \in [0, \widehat{t}]$, using an exponential distribution as an example. Differentiating (19), I can write:

$$\begin{aligned}\frac{\partial LHS(A)}{\partial A} - \frac{\partial RHS(A)}{\partial A} &= 1 - G(\alpha_S) + \frac{\gamma}{2\eta M_E} - g(\alpha_S)\frac{\partial \alpha_S}{\partial A}A - \frac{\partial \overline{p}_u}{\partial A} \\ &= \left(1 - G(\alpha_S) - \frac{\partial \overline{p}_u^{direct}}{\partial A}\right) + \frac{\gamma}{2\eta M_E} - g(\alpha_S)\frac{\partial \alpha_S}{\partial A}A - \frac{\partial \overline{p}_u^{indirect}}{\partial A}\end{aligned}$$

where,

$$\frac{\partial \overline{p}_u}{\partial A} = \frac{\partial \overline{p}_u^{direct}}{\partial A} + \frac{\partial \overline{p}_u^{indirect}}{\partial A}$$

I am splitting up $\frac{\partial \overline{p}_u}{\partial A}$ via Leibniz rule. The term $\frac{\partial \overline{p}_u^{direct}}{\partial A}$ contains the effects of A from inside the integrand. The term $\frac{\partial \overline{p}_u^{indirect}}{\partial A}$ are the effects of A through the limits of integration. To show that $\frac{\partial LHS(A)}{\partial A} - \frac{\partial RHS(A)}{\partial A} > 0$, I will first show that $\left(1 - G(\alpha_S) - \frac{\partial \overline{p}_u^{direct}}{\partial A}\right) > 0$. For $t < \underline{t}(\delta)$, this term can be written as:

$$\begin{aligned}\left(1 - G(\alpha_S) - \frac{\partial \overline{p}_u^{direct}}{\partial A}\right) &= \int_{\alpha_S}^{\underline{\alpha}_B} \left(1 - \frac{\frac{\partial p_N^H}{\partial A} + \frac{\partial p_N^X}{\partial A}}{2}\right) dG + \int_{\underline{\alpha}_B}^{\overline{\alpha}_B} \left(1 - \frac{\frac{\partial P_B^H}{\partial A} + \frac{\partial P_B^X}{\partial A}}{2}\right) dG \\ &\quad + \int_{\overline{\alpha}_B}^{\infty} \left(1 - \frac{\frac{\partial p_N^H}{\partial A} + \frac{\partial p_N^X}{\partial A}}{2}\right) dG \\ &> 0\end{aligned}$$

where $\frac{\partial p_N^H}{\partial A} + \frac{\partial p_N^X}{\partial A} < 2$, $\frac{\partial p_N^H}{\partial A} + \frac{\partial p_N^X}{\partial A} < 2$, and $\frac{\partial p_N^H}{\partial A} + \frac{\partial p_N^X}{\partial A} < 2$ all follow trivially form the fact that $b > 0$ (demand is not perfect elastic). Similar analysis follows for $\underline{t}(\delta) < t < \overline{t}(\delta)$, and $t > \overline{t}(\delta)$. To finish the proof of uniqueness, I need to show that:

$$g(\alpha_S)\frac{\partial \alpha_S}{\partial A}A + \frac{\partial \overline{p}_u^{indirect}}{\partial A} < \left(1 - G(\alpha_S) - \frac{\partial \overline{p}_u^{direct}}{\partial A}\right) + \frac{\gamma}{2\eta M_E}$$

Analytically, this is a not possible for all values of t. Thus, I will show how D1 is sufficient for uniqueness. The analysis is similar for all $t \in [0, \widehat{t}]$, and I will present the proof for $\underline{t}(\delta) < t < \overline{t}(\delta)$.

For $\underline{t}(\delta) < t < \overline{t}(\delta)$, I can write:

$$\begin{aligned}g(\alpha_S)\frac{\partial \alpha_S}{\partial A}A + \frac{\partial \overline{p}_u^{indirect}}{\partial A} &= \left(A - (P_N^H(\alpha_S) + P_N^X(\alpha_S))/2\right) g(\alpha_S)\frac{\partial \alpha_S}{\partial A} &(20)\\ &\quad + g(\underline{\alpha}_B)\frac{\partial \underline{\alpha}_B}{\partial A}\left(P_N^H(\underline{\alpha}_B) + P_N^X(\underline{\alpha}_B) - P_B^H(\underline{\alpha}_B) - P_B^X(\underline{\alpha}_B)\right)/2 &(21)\\ &\quad + g(\alpha_{BB^*})\frac{\partial \alpha_{BB^*}}{\partial A}\left((P_B^H(\alpha_{BB^*}) + P_B^X(\alpha_{BB^*}))/2 - P^{B^*}(\alpha_{BB^*})\right) \\ &\quad + g(\overline{\alpha}_B^*)\frac{\partial \overline{\alpha}_B^*}{\partial A}\left(P^{B^*}(\overline{\alpha}_B) - P_N^H(\overline{\alpha}_B) - P_N^X(\overline{\alpha}_B)\right)/2\end{aligned}$$

The strategy for uniqueness is to show that $g(\alpha_S)\frac{\partial \alpha_S}{\partial A}$, $g(\underline{\alpha}_B)\frac{\partial \underline{\alpha}_B}{\partial A}$, $g(\alpha_{BB^*})\frac{\partial \alpha_{BB^*}}{\partial A}$ and $g(\overline{\alpha}^*_B)\frac{\partial \overline{\alpha}^*_B}{\partial A}$ are small enough. If this is the case, uniqueness is proven by ensuring that the term $g(\alpha_S)\frac{\partial \alpha_S}{\partial A}A + \frac{\partial \overline{p}_u^{indirect}}{\partial A}$ is sufficiently small. To derive conditions such that this is the case, I must first solve for $\frac{\partial \alpha_S}{\partial A}$, $\frac{\partial \underline{\alpha}_B}{\partial A}$, $\frac{\partial \overline{\alpha}^*_B}{\partial A}$ and $\frac{\partial \alpha_{BB^*}}{\partial A}$.

It can be shown that $\frac{\partial \alpha_{BB^*}}{\partial A}$ is the only derivative that can be unambiguously signed, as it does not enter into the acquisition market clearing condition. This is written as:

$$\frac{\partial \alpha_{BB^*}}{\partial A} = -\frac{\frac{\partial BB^*(\alpha_{BB^*})}{\partial A}}{\frac{\partial BB^*(\alpha_{BB^*})}{\partial \alpha}} < 0$$

Since prices are lower if a firm avoids trade costs, $\left((P_B^H(\alpha_{BB^*}) + P_B^X(\alpha_{BB^*}))/2 - P^{B^*}(\alpha_{BB^*})\right) > 0$. This implies that the term $g(\alpha_{BB^*})\frac{\partial \alpha_{BB^*}}{\partial A}\left((P_B^H(\alpha_{BB^*}) + P_B^X(\alpha_{BB^*}))/2 - P^{B^*}(\alpha_{BB^*})\right)$ is negative.

The terms $\frac{\partial \alpha_S}{\partial A}$, $\frac{\partial \underline{\alpha}_B}{\partial A}$, $\frac{\partial \overline{\alpha}^*_B}{\partial A}$ are derived using the following:

$$\begin{bmatrix} -\frac{\partial \pi^N(\alpha_S)}{\partial \alpha} & \frac{\partial \Delta\Pi(\underline{\alpha}_B)}{\partial \alpha} & 0 \\ -\frac{\partial \pi^N(\alpha_S)}{\partial \alpha} & 0 & \frac{\partial \Delta\Pi^*(\overline{\alpha}^*_B)}{\partial \alpha} \\ g(\alpha_S) & g(\underline{\alpha}_B) & -g(\overline{\alpha}^*_B) \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \alpha_S}{\partial A} \\ \frac{\partial \underline{\alpha}_B}{\partial A} \\ \frac{\partial \overline{\alpha}^*_B}{\partial A} \end{bmatrix} = \begin{bmatrix} \frac{\partial \pi^N(\alpha_S)}{\partial A} - \frac{\partial \Delta\Pi(\underline{\alpha}_B)}{\partial A} \\ \frac{\partial \pi^N(\alpha_S)}{\partial A} - \frac{\partial \Delta\Pi^*(\overline{\alpha}^*_B)}{\partial A} \\ 0 \end{bmatrix}$$

Solving this system of equations, we get:

$$\begin{aligned} \frac{\partial \alpha_S}{\partial A} &= \left(-\frac{1}{D}\right)\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial \alpha} g(\underline{\alpha}_B)\left(\frac{\partial \pi^N\left(\alpha_S\right)}{\partial A} - \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial A}\right) - \left(-\frac{1}{D}\right)\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha} g(\overline{\alpha}^*_B)\underbrace{\left(\frac{\partial \pi^N\left(\alpha_S\right)}{\partial A} - \frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial A}\right)}_{?} \\ &\lesseqgtr 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial \underline{\alpha}_B}{\partial A} &= \left(-\frac{1}{D}\right)\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\overline{\alpha}^*_B)\underbrace{\left(\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial A} - \frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial A}\right)}_{?} + \left(-\frac{1}{D}\right)\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial \alpha} g(\alpha_S)\left(\frac{\partial \pi^N\left(\alpha_S\right)}{\partial A} - \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial A}\right) \\ &\lesseqgtr 0 \end{aligned}$$

$$\frac{\partial \alpha_{BB^*}}{\partial A} = -\frac{\frac{\partial BB^*(\alpha_{BB^*})}{\partial A}}{\frac{\partial BB^*(\alpha_{BB^*})}{\partial \alpha}} < 0$$

$$\begin{aligned} \frac{\partial \overline{\alpha}^*_B}{\partial A} &= \left(-\frac{1}{D}\right)\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\underline{\alpha}_B)\underbrace{\left(\frac{\partial \Delta\Pi\left(\overline{\alpha}^*_B\right)}{\partial A} - \frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B\right)}{\partial A}\right)}_{?} + \left(-\frac{1}{D}\right)\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha} g(\alpha_S)\underbrace{\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial A} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)}_{?} \\ &\lesseqgtr 0 \end{aligned}$$

where:

$$\begin{aligned} D &= \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha}\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial \alpha} g(\alpha_S) - \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha}\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\overline{\alpha}^*_B) + \frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial \alpha}\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\underline{\alpha}_B) \\ &< 0 \end{aligned}$$

Writing $g(\alpha_S)\frac{\partial\alpha_S}{\partial A}$:

$$g(\alpha_S)\frac{\partial\alpha_S}{\partial A} = \frac{g(\alpha_S)g(\underline{\alpha}_B)\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\left(\frac{\partial\pi^N(\alpha_S)}{\partial A}-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}\right)}{-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}g(\alpha_S)+\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}g(\overline{\alpha}^*_B)-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}g(\underline{\alpha}_B)}$$
$$-\frac{g(\alpha_S)g(\overline{\alpha}^*_B)\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\left(\frac{\partial\pi^N(\alpha_S)}{\partial A}-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial A}\right)}{-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}g(\alpha_S)+\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}g(\overline{\alpha}^*_B)-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}g(\underline{\alpha}_B)} \quad (22)$$

Generally, as $g(\alpha)$ becomes small for all productivity cutoffs, the numerator of (22) will approach zero faster than the denominator of (22). This is also the case for $g(\underline{\alpha}_B)\frac{\partial\underline{\alpha}_B}{\partial A}$ and $g(\overline{\alpha}^*_B)\frac{\partial\overline{\alpha}^*_B}{\partial A}$. As a concrete example, consider the exponential distribution, where $g(\alpha)=\lambda e^{-\lambda\alpha}$. Defining $\widetilde{g}(\alpha)=e^{-\lambda\alpha}$, I can write (22) as:

$$g(\alpha_S)\frac{\partial\alpha_S}{\partial A} = \frac{\lambda\widetilde{g}(\alpha_S)\widetilde{g}(\underline{\alpha}_B)\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\left(\frac{\partial\pi^N(\alpha_S)}{\partial A}-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}\right)}{-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\widetilde{g}(\alpha_S)+\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\overline{\alpha}^*_B)-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\underline{\alpha}_B)}+$$
$$-\frac{\lambda\widetilde{g}(\alpha_S)\widetilde{g}(\overline{\alpha}^*_B)\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\left(\frac{\partial\pi^N(\alpha_S)}{\partial A}-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial A}\right)}{-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\widetilde{g}(\alpha_S)+\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\overline{\alpha}^*_B)-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\underline{\alpha}_B)} \quad (23)$$

Noting that $\lim_{\lambda\to 0}\widetilde{g}(\alpha)=1$ for all α, it is immediate that $\lim_{\lambda\to 0}\left(g(\alpha_S)\frac{\partial\alpha_S}{\partial A}\right)=0$. I can write similar expressions for $g(\underline{\alpha}_B)\frac{\partial\underline{\alpha}_B}{\partial A}$, $g(\overline{\alpha}^*_B)\frac{\partial\overline{\alpha}^*_B}{\partial A}$ and $g(\alpha_{BB^*})\frac{\partial\alpha_{BB^*}}{\partial A}$:

$$g(\underline{\alpha}_B)\frac{\partial\underline{\alpha}_B}{\partial A} = \frac{\lambda\widetilde{g}(\underline{\alpha}_B)\widetilde{g}(\overline{\alpha}^*_B)\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\left(\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial A}\right)}{-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\widetilde{g}(\alpha_S)+\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\overline{\alpha}^*_B)-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\underline{\alpha}_B)} \quad (24)$$
$$+\frac{\lambda\widetilde{g}(\underline{\alpha}_B)\widetilde{g}(\alpha_S)\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\left(\frac{\partial\pi^N(\alpha_S)}{\partial A}-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}\right)}{-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\widetilde{g}(\alpha_S)+\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\overline{\alpha}^*_B)-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\underline{\alpha}_B)}$$

$$g(\overline{\alpha}^*_B)\frac{\partial\alpha_{BB^*}}{\partial A} = -\lambda\widetilde{g}(\alpha_{BB^*})\frac{\frac{\partial BB^*(\alpha_{BB^*})}{\partial A}}{\frac{\partial BB^*(\alpha_{BB^*})}{\partial\alpha}} \quad (25)$$

$$g(\alpha_{BB^*})\frac{\partial\overline{\alpha}^*_B}{\partial A} = \frac{\lambda\widetilde{g}(\overline{\alpha}^*_B)\widetilde{g}(\underline{\alpha}_B)\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\left(\frac{\partial\Delta\Pi(\overline{\alpha}^*_B)}{\partial A}-\frac{\partial\Delta\Pi^*(\underline{\alpha}_B)}{\partial A}\right)}{-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\widetilde{g}(\alpha_S)+\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\overline{\alpha}^*_B)-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\underline{\alpha}_B)} \quad (26)$$
$$+\frac{\lambda\widetilde{g}(\overline{\alpha}^*_B)\widetilde{g}(\alpha_S)\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\left(\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial A}-\frac{\partial\pi^N(\alpha_S)}{\partial A}\right)}{-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\widetilde{g}(\alpha_S)+\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\overline{\alpha}^*_B)-\frac{\partial\Delta\Pi^*(\overline{\alpha}^*_B)}{\partial\alpha}\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}\widetilde{g}(\underline{\alpha}_B)}$$

Clearly, $g(\underline{\alpha}_B)\frac{\partial\underline{\alpha}_B}{\partial A}$, $g(\overline{\alpha}^*_B)\frac{\partial\overline{\alpha}^*_B}{\partial A}$ and $g(\alpha_{BB^*})\frac{\partial\alpha_{BB^*}}{\partial A}$ all approach zero as $\lambda\to 0$.

Using (23), (24), (25) and (26), it is immediate that $\lim_{\lambda\to 0}\left(g(\alpha_S)\frac{\partial\alpha_S}{\partial A}A+\frac{\partial\overline{p}_u^{indirect}}{\partial A}\right)<0$. Thus, there must exist a λ small enough such that,

$$g(\alpha_S)\frac{\partial\alpha_S}{\partial A}A+\frac{\partial\overline{p}_u^{indirect}}{\partial A}<\left(1-G\left(\alpha_S\right)-\frac{\partial\overline{p}_u^{direct}}{\partial A}\right)+\frac{\gamma}{2\eta M_E}$$

which guarantees a unique equilibrium A. This analysis can also extend to a Pareto distribution, provided the that lower bound on productivity is not binding. ■

Free Entry

To close the open economy equilibrium, I now define the free entry condition. After imposing the acquisition market clearing condition, the free entry condition is written as follows:

$$\int_{\alpha\in\Theta_N}\pi^N\left(\alpha\right)dG\left(\alpha\right)+\int_{\alpha\in\Theta_B}\pi^B\left(\alpha\right)dG\left(\alpha\right)+\int_{\alpha\in\Theta_{B^*}}\left(\pi^{B^*}\left(\alpha\right)-\delta\right)dG\left(\alpha\right)=F_E \tag{27}$$

With D1, it is guaranteed that additional entry lowers the ex-ante profitability for all entrants. Thus, provided that the fixed cost of entry is not prohibitively high, there will exist a unique number of entering firms.

1.1 Proof of Proposition 3

$t<\underline{t}(\delta)$

When differentiating any free entry condition with respect to t, the envelope theorem will eliminate any indirect effects through changes in our productivity cutoffs. Differentiating (27) with respect to t, we get:

$$\frac{\partial A}{\partial t}=-\frac{\left(\int_{\alpha_S}^{\underline{\alpha}_B}\frac{\partial\pi^{N,STC}(\alpha)}{\partial t}dG\left(\alpha\right)+2\int_{\underline{\alpha}_B}^{\overline{\alpha}_B}\frac{\partial\pi^{B,STC}(\alpha)}{\partial t}dG\left(\alpha\right)+\int_{\overline{\alpha}_B}^{\infty}\frac{\partial\pi^{N,STC}(\alpha)}{\partial t}dG\left(\alpha\right)\right)}{\left(\int_{\alpha_S}^{\underline{\alpha}_B}\frac{\partial\pi^{N,STC}(\alpha)}{\partial A}dG\left(\alpha\right)+2\int_{\underline{\alpha}_B}^{\overline{\alpha}_B}\frac{\partial\pi^{B,STC}(\alpha)}{\partial A}dG\left(\alpha\right)+\int_{\overline{\alpha}_B}^{\infty}\frac{\partial\pi^{N,STC}(\alpha)}{\partial A}dG\left(\alpha\right)\right)}<1$$

where this derivative is less than one via $\frac{\partial\pi^{j,STC}(\alpha)}{\partial A}>-\frac{\partial\pi^{j,STC}(\alpha)}{\partial t}>0$ for all acquisition outcomes j.

Differentiating the equilibrium conditions in (5), (8), (7), (27), and the acquisition market clearing condition for $t<\underline{t}(\delta)$, we get:

$$\begin{bmatrix}\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}-\frac{\partial\pi^N(\alpha_S)}{\partial A} & -\frac{\partial\pi^N(\alpha_S)}{\partial\alpha} & \frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha} & 0\\ \frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial A}-\frac{\partial\pi^N(\alpha_S)}{\partial A} & -\frac{\partial\pi^N(\alpha_S)}{\partial\alpha} & 0 & \frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial\alpha}\\ \frac{\partial E\pi}{\partial A} & 0 & 0 & 0\\ 0 & g(\alpha_S) & g(\underline{\alpha}_B) & -g(\overline{\alpha}_B)\end{bmatrix}\begin{bmatrix}\frac{\partial A}{\partial t}\\ \frac{\partial\alpha_S}{\partial t}\\ \frac{\partial\underline{\alpha}_B}{\partial t}\\ \frac{\partial\overline{\alpha}_B}{\partial t}\end{bmatrix}=\begin{bmatrix}\frac{\partial\pi^N(\alpha_S)}{\partial t}-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial t}\\ \frac{\partial\pi^N(\alpha_S)}{\partial t}-\frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial t}\\ -\frac{\partial E\pi}{\partial t}\\ 0\end{bmatrix}$$

noting that,

$$\begin{aligned}\frac{\partial\pi^N\left(\alpha_S\right)}{\partial A}-\frac{\partial\Delta\Pi\left(\underline{\alpha}_B\right)}{\partial A} &= \frac{\partial\pi^N\left(\alpha_S\right)}{\partial A}-\frac{\partial\Delta\Pi\left(\overline{\alpha}_B\right)}{\partial A}=-\frac{t^2}{2bk\left(2A-t\right)}\\ \frac{\partial\pi^N\left(\alpha_S\right)}{\partial t}-\frac{\partial\Delta\Pi\left(\underline{\alpha}_B\right)}{\partial t} &= \frac{\partial\pi^N\left(\alpha_S\right)}{\partial t}-\frac{\partial\Delta\Pi\left(\overline{\alpha}_B\right)}{\partial t}=\frac{At}{2bk\left(2A-t\right)}\end{aligned}$$

which yields,

$$\frac{\partial\pi^N(\alpha_S)}{\partial t}-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial t}>\frac{\partial\pi^N(\alpha_S)}{\partial A}-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}>\frac{\partial A}{\partial t}\left(\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}-\frac{\partial\pi(\alpha_S)}{\partial A}\right)$$

and

$$\frac{\partial\pi^N(\alpha_S)}{\partial t}-\frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial t}>\frac{\partial\pi^N(\alpha_S)}{\partial A}-\frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial A}>\frac{\partial A}{\partial t}\left(\frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial A}-\frac{\partial\pi(\alpha_S)}{\partial A}\right)$$

we can solve the system of equations and get:

$$\begin{aligned}\frac{\partial\alpha_S}{\partial t} &= \left(\frac{1}{D}\right)\left(\overbrace{-\frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial\alpha}}^{+}g(\underline{\alpha}_B)+\overbrace{\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}}^{+}g(\overline{\alpha}_B)\right)\left(\frac{\partial\pi(\alpha_S)}{\partial t}-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial t}-\frac{\partial A}{\partial t}\left(\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}-\frac{\partial\pi(\alpha_S)}{\partial A}\right)\right)\\ &< 0\end{aligned}$$

$$\begin{aligned}\frac{\partial\underline{\alpha}_B}{\partial t} &= \left(\frac{1}{D}\right)\left(\overbrace{\frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial\alpha}}^{-}g(\alpha_S)\right)\left(\frac{\partial\pi(\alpha_S)}{\partial t}-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial t}-\frac{\partial A}{\partial t}\left(\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}-\frac{\partial\pi(\alpha_S)}{\partial A}\right)\right)\\ &> 0\end{aligned}$$

$$\begin{aligned}\frac{\partial\overline{\alpha}_B}{\partial t} &= \left(\frac{1}{D}\right)\left(\overbrace{\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}}^{+}g(\alpha_S)\right)\left(\frac{\partial\pi(\alpha_S)}{\partial t}-\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial t}-\frac{\partial A}{\partial t}\left(\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial A}-\frac{\partial\pi(\alpha_S)}{\partial A}\right)\right)\\ &< 0\end{aligned}$$

where:

$$\begin{aligned}D &= \overbrace{\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}}^{+}\overbrace{\frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial\alpha}}^{-}g(\alpha_S)-\overbrace{\frac{\partial\Delta\Pi(\underline{\alpha}_B)}{\partial\alpha}}^{+}\overbrace{\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}}^{+}g(\overline{\alpha}_B)+\overbrace{\frac{\partial\Delta\Pi(\overline{\alpha}_B)}{\partial\alpha}}^{-}\overbrace{\frac{\partial\pi^N(\alpha_S)}{\partial\alpha}}^{+}g(\underline{\alpha}_B)\\ &< 0\end{aligned}$$

1.2 Proof of Proposition 4

$\underline{t}(\delta)<t<\overline{t}(\delta)$

Similar to above, I can write $\frac{\partial A}{\partial t}$ as:

$$\frac{\partial A}{\partial t}=-\frac{\left(\int_{\alpha_S}^{\underline{\alpha}_B}\frac{\partial\pi^{N,STC}(\alpha)}{\partial t}dG(\alpha)+2\int_{\underline{\alpha}_B}^{\alpha_{BB^*}}\frac{\partial\pi^{B,STC}(\alpha)}{\partial t}dG(\alpha)+\int_{\overline{\alpha}_B^*}^{\infty}\frac{\partial\pi^{N,STC}(\alpha)}{\partial t}dG(\alpha)\right)}{\left(\int_{\alpha_S}^{\underline{\alpha}_B}\frac{\partial\pi^{N,STC}(\alpha)}{\partial A}dG(\alpha)+2\int_{\underline{\alpha}_B}^{\alpha_{BB^*}}\frac{\partial\pi^{B,STC}(\alpha)}{\partial A}dG(\alpha)+\int_{\alpha_{BB^*}}^{\overline{\alpha}_B^*}\frac{\partial\pi^{F,STC}(\alpha)}{\partial A}dG(\alpha)+\int_{\overline{\alpha}_B^*}^{\infty}\frac{\partial\pi^{N,STC}(\alpha)}{\partial A}dG(\alpha)\right)}<1$$

where this derivative is less than zero via $\frac{\partial\pi^{j,STC}(\alpha)}{\partial A}>-\frac{\partial\pi^{j,STC}(\alpha)}{\partial t}$ for all acquisition outcomes j.

Differentiating the equilibrium conditions in (5), (11), (7), (27), and the acquisition market clearing condition for $\underline{t}(\delta)<t<\overline{t}(\delta)$, we get:

$$\begin{bmatrix} \frac{\partial \Delta\Pi(\underline{\alpha}_B)}{\partial A} - \frac{\partial \pi^N(\alpha_S)}{\partial A} & -\frac{\partial \pi^N(\alpha_S)}{\partial \alpha} & \frac{\partial \Delta\Pi(\underline{\alpha}_B)}{\partial \alpha} & 0 \\ \frac{\partial \Delta\Pi^*(\overline{\alpha}^*_B)}{\partial A} - \frac{\partial \pi^N(\alpha_S)}{\partial A} & -\frac{\partial \pi^N(\alpha_S)}{\partial \alpha} & 0 & \frac{\partial \Delta\Pi^*(\overline{\alpha}^*_B)}{\partial \alpha} \\ \frac{\partial E\pi}{\partial A} & 0 & 0 & 0 \\ 0 & g(\alpha_S) & g(\underline{\alpha}_B) & -g(\overline{\alpha}^*_B) \end{bmatrix} \begin{bmatrix} \frac{\partial A}{\partial t} \\ \frac{\partial \alpha_S}{\partial t} \\ \frac{\partial \underline{\alpha}_B}{\partial t} \\ \frac{\partial \overline{\alpha}^*_B}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial \pi^N(\alpha_S)}{\partial t} - \frac{\partial \Delta\Pi(\underline{\alpha}_B)}{\partial t} \\ \frac{\partial \pi^N(\alpha_S)}{\partial t} - \frac{\partial \Delta\Pi^*(\overline{\alpha}^*_B)}{\partial t} \\ -\frac{\partial E\pi}{\partial t} \\ 0 \end{bmatrix}$$

Continuing, we can write:

$$\begin{aligned} \frac{\partial \alpha_S}{\partial t} &= -\frac{1}{D}\overbrace{\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha} g(\overline{\alpha}^*_B)\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial t} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial t} + \frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial A} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)\right)}^{>0} \\ &\quad -\frac{1}{D}\overbrace{\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial \alpha} g(\underline{\alpha}_B)\left(\frac{\partial \pi^N\left(\alpha_S\right)}{\partial t} - \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial t} - \frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial A} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)\right)}^{<0} \\ &\lesseqgtr 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial \underline{\alpha}_B}{\partial t} &= -\frac{1}{D}\overbrace{\frac{\partial \overset{+}{\pi^N}\left(\alpha_S\right)}{\partial \alpha} g(\overline{\alpha}^*_B)\left(\frac{\partial \overset{+}{\Delta\Pi^*}\left(\overline{\alpha}^*_B\right)}{\partial t} - \frac{\partial \overset{-}{\pi^N}\left(\alpha_S\right)}{\partial t} + \frac{\partial \overset{+}{A}}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial A} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)\right)}^{>0 \text{ (see below)}} \\ &\quad -\frac{1}{D}\overbrace{\left(\frac{\partial \overset{+}{\pi^N}\left(\alpha_S\right)}{\partial \alpha} g(\overline{\alpha}^*_B) - \frac{\partial \overset{-}{\Delta\Pi^*}\left(\overline{\alpha}^*_B\right)}{\partial \alpha} g(\alpha_S)\right)\left(\overbrace{\frac{\partial \pi^N\left(\alpha_S\right)}{\partial t} - \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial t} - \frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial A} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)}^{+}\right)}^{>0} \\ &> 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial \overline{\alpha}^*_B}{\partial t} &= -\frac{1}{D}\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\underline{\alpha}_B)\overbrace{\left(\frac{\partial \pi^N\left(\alpha_S\right)}{\partial t} - \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial t} - \frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial A} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)\right)}^{>0} \\ &\quad -\frac{1}{D}\overbrace{\left(\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha} g(\alpha_S) + \frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\underline{\alpha}_B)\right)\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial t} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial t} + \frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial A} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)\right)}^{>0} \\ &> 0 \end{aligned}$$

where:

$$\begin{aligned} D &= \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha}\frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial \alpha} g(\alpha_S) - \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha}\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\overline{\alpha}^*_B) + \frac{\partial \Delta\Pi^*\left(\overline{\alpha}^*_B\right)}{\partial \alpha}\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\underline{\alpha}_B) \\ &< 0 \end{aligned}$$

and,

$$\begin{aligned} \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A} - \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial A} &= -\frac{t^2}{2bk\left(2A - t\right)} \\ \frac{\partial \pi^N\left(\alpha_S\right)}{\partial t} - \frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial t} &= \frac{At}{2bk\left(2A - t\right)} \end{aligned}$$

The sign of $\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t} - \frac{\partial \pi^N(\alpha_S)}{\partial t} + \frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A} - \frac{\partial \pi^N(\alpha_S)}{\partial A}\right)$ is not totally straightforward. While it is clear that $\overset{+}{\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t}} - \overset{-}{\frac{\partial \pi^N(\alpha_S)}{\partial t}} > 0$, the sign of $\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A} - \frac{\partial \pi^N(\alpha_S)}{\partial A}\right)$ is ambiguous. However, since $\frac{\partial A}{\partial t} \in (0,1)$, if we can show that $\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t} - \frac{\partial \pi^N(\alpha_S)}{\partial t} + \left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A} - \frac{\partial \pi^N(\alpha_S)}{\partial A}\right) > 0$, it must be the case that $\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t} - \frac{\partial \pi^N(\alpha_S)}{\partial t} + \frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A} - \frac{\partial \pi^N(\alpha_S)}{\partial A}\right) > 0$. As a function of model parameters, we can write:

$$\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial t} + \left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right) = \frac{(A-t)(2(A-t)b\overline{\alpha}_B^* k - vt)}{\left(2b\overline{\alpha}_B^* k + v\right)(2A-t)} + \frac{2\delta}{(2A-t)}$$

Since $(2(A-t)b\overline{\alpha}_B^* k - vt) > 0$ for firms that are exporters, and $t < A$ for any export status to occur, it is clear that $\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t} - \frac{\partial \pi^N(\alpha_S)}{\partial t} + \left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A} - \frac{\partial \pi^N(\alpha_S)}{\partial A}\right) > 0$. Thus,

$$\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial t} + \frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A} - \frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right) > 0$$

Finally, differentiating (13),

$$\frac{\partial \alpha_{BB^*}}{\partial t} = -\frac{\overbrace{\left(\frac{\partial \Delta\Pi^*\left(\alpha_{BB^*}\right)}{\partial A} - \frac{\partial \Delta\Pi\left(\alpha_{BB^*}\right)}{\partial A}\right)}^{>0}\frac{\partial A}{\partial t}}{\underbrace{\left(\frac{\partial \Delta\Pi^*\left(\alpha_{BB^*}\right)}{\partial \alpha} - \frac{\partial \Delta\Pi\left(\alpha_{BB^*}\right)}{\partial \alpha}\right)}_{>0}} < 0$$

1.3 Proof of Proposition 5

$\overline{t}(\delta) < t$

Virtually identical to above, we can write $\frac{\partial A}{\partial t}$ as:

$$\frac{\partial A}{\partial t} = -\frac{\left(\int_{\alpha_S}^{\underline{\alpha}_B} \frac{\partial \pi^{N,STC}(\alpha)}{\partial t} dG\left(\alpha\right) + \int_{\overline{\alpha}_B^*}^{\infty} \frac{\partial \pi^{N,STC}(\alpha)}{\partial t} dG\left(\alpha\right)\right)}{\left(\int_{\alpha_S}^{\underline{\alpha}_B} \frac{\partial \pi^{N,STC}(\alpha)}{\partial A} dG\left(\alpha\right) + \int_{\underline{\alpha}_B^*}^{\overline{\alpha}_B^*} \frac{\partial \pi^{F,STC}(\alpha)}{\partial A} dG\left(\alpha\right) + \int_{\overline{\alpha}_B^*}^{\infty} \frac{\partial \pi^{N,STC}(\alpha)}{\partial A} dG\left(\alpha\right)\right)} < 1$$

where this derivative is less than zero via $\frac{\partial \pi^{j,STC}(\alpha)}{\partial A} > -\frac{\partial \pi^{j,STC}(\alpha)}{\partial t}$ for all acquisition outcomes j.

Differentiating the equilibrium conditions in (5), (11), (10), (27), and the acquisition market clearing condition for $\overline{t}(\delta) < t$, we get:

$$\begin{bmatrix} \frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right)}{\partial A} - \frac{\partial \pi^N(\alpha_S)}{\partial A} & -\frac{\partial \pi^N(\alpha_S)}{\partial \alpha} & \frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right)}{\partial \alpha} & 0 \\ \frac{\partial \Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial \pi^N(\alpha_S)}{\partial A} & -\frac{\partial \pi^N(\alpha_S)}{\partial \alpha} & 0 & \frac{\partial \Delta\Pi^*(\overline{\alpha}_B^*)}{\partial \alpha} \\ \frac{\partial E\pi}{\partial A} & 0 & 0 & 0 \\ 0 & g(\alpha_S) & g(\underline{\alpha}_B^*) & -g(\overline{\alpha}_B^*) \end{bmatrix} \begin{bmatrix} \frac{\partial A}{\partial t} \\ \frac{\partial \alpha_S}{\partial t} \\ \frac{\partial \underline{\alpha}_B^*}{\partial t} \\ \frac{\partial \overline{\alpha}_B^*}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial \pi^N(\alpha_S)}{\partial t} - \frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B\right)}{\partial t} \\ \frac{\partial \pi^N(\alpha_S)}{\partial t} - \frac{\partial \Delta\Pi^*(\overline{\alpha}_B^*)}{\partial t} \\ -\frac{\partial E\pi}{\partial t} \\ 0 \end{bmatrix}$$

Continuing, we can write (for signs of each term, see below):

$$\begin{aligned}
\frac{\partial \alpha_S}{\partial t} &= -\frac{1}{D}\overbrace{\left(-\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial \alpha}\right) g(\underline{\alpha}_B^*)\left(\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial t}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial t}+\frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial A}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)\right)}^{>0} \\
&\quad -\frac{1}{D}\overbrace{\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha} g(\overline{\alpha}_B^*)\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial t}+\frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)\right)}^{>0} \\
&> 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \underline{\alpha}_B^*}{\partial t} &= \frac{1}{D}\overbrace{\left(-\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial \alpha} g(\alpha_S)\right)\left(\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial t}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial t}+\frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial A}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)\right)}^{>0} \\
&\quad -\frac{1}{D}\overbrace{\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\overline{\alpha}_B^*)\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t}-\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial t}+\frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A}-\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial A}\right)\right)}^{>0} \\
&\lesseqgtr 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \overline{\alpha}_B^*}{\partial t} &= -\frac{1}{D}\overbrace{\frac{\partial \Delta\Pi\left(\underline{\alpha}_B\right)}{\partial \alpha} g(\alpha_S)\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial t}+\frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right)\right)}^{>0} \\
&\quad -\frac{1}{D}\overbrace{\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\underline{\alpha}_B)\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t}-\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial t}+\frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A}-\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial A}\right)\right)}^{>0} \\
&> 0
\end{aligned}$$

where:

$$\begin{aligned}
D &= \frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right)}{\partial \alpha}\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial \alpha} g(\alpha_S)-\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right)}{\partial \alpha}\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\overline{\alpha}_B^*)+\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial \alpha}\frac{\partial \pi^N\left(\alpha_S\right)}{\partial \alpha} g(\underline{\alpha}_B) \\
&< 0
\end{aligned}$$

By similar work to above, I can show:

$$\begin{aligned}
\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial t}+\frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right) &> 0 \\
\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial t}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial t}+\frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial A}-\frac{\partial \pi^N\left(\alpha_S\right)}{\partial A}\right) &> 0
\end{aligned}$$

Finally, I must sign $\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t}-\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial t}+\frac{\partial A}{\partial t}\left(\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A}-\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial A}\right)$. First, I will establish that $\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial t}-\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial t} > 0$ and $\frac{\partial \Delta\Pi^*\left(\overline{\alpha}_B^*\right)}{\partial A}-\frac{\partial \Delta\Pi^*\left(\underline{\alpha}_B^*\right))}{\partial A} < 0$. For any value of α, $\frac{\partial \Delta\Pi^*(\alpha)}{\partial t}$ and $\frac{\partial \Delta\Pi^*(\alpha)}{\partial A}$

are written as:

$$\begin{aligned}\frac{\partial\Delta\Pi^*(\alpha)}{\partial t} &= \frac{(2bA\alpha k - 2b\alpha kt - vt)}{4b(b\alpha k + v)} > 0 \textit{ (for exporters)} \\ \frac{\partial\Delta\Pi^*(\alpha)}{\partial A} &= \frac{\alpha k(2Av + 2b\alpha kt + vt)}{2(b\alpha k + v)(2b\alpha k + v)} > 0\end{aligned}$$

To sign $\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial t} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial t} > 0$ and $\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial A} < 0$, I also need to define $\underline{\alpha}_B^*(\overline{\alpha}_B^*)$:

$$\underline{\alpha}_B^*(\overline{\alpha}_B^*) = \frac{v(8tbAk\overline{\alpha}_B^* - 2bkt^2\overline{\alpha}_B^* + 4A^2v + 4tAv - t^2v)}{2bk(4A^2\overline{\alpha}_B^*kb + 2bkt^2\overline{\alpha}_B^* - 8tbAk\overline{\alpha}_B^* + t^2v - 4tAv)}$$

With $\underline{\alpha}_B^*(\overline{\alpha}_B^*)$, I can write:

$$\begin{aligned}\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial A} &= -\frac{(8A^2b^2\overline{\alpha}_B^{*2}k^2 - 4A^2v^2 - 16tb^2A\overline{\alpha}_B^{*2}k^2 - 16tbA\overline{\alpha}_B^*kv)t}{4A(2A-t)(2b\overline{\alpha}_B^*k + v)b(b\overline{\alpha}_B^*k + v)} \\ &\quad -\frac{(4b^2\overline{\alpha}_B^{*2}k^2t^2 + 4b\overline{\alpha}_B^*kt^2v + v^2t^2 - 4tAv^2)t}{4A(2A-t)(2b\overline{\alpha}_B^*k + v)b(b\overline{\alpha}_B^*k + v)} \\ \frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial t} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial t} &= -\frac{A}{t}\left(\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial A}\right)\end{aligned}$$

Since $A > t$, then

$$\left|\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial t} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial t}\right| > \left|\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial A}\right|$$

Also,

$$sign\left(\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial t} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial t}\right) = -sign\left(\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial A}\right)$$

Thus, since $\frac{\partial A}{\partial t} \in (0,1)$, if $\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial A} < 0$, then we can sign the derivative in question. Continuing, $\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial A} < 0$ if

$$\overline{\alpha}_B^* > \frac{\sqrt{2}v(2A + t + \sqrt{2}t)}{2b(2A - 2t - \sqrt{2}t)k}$$

In Lemma 2, I establish that this condition holds, as the maximum of $\Delta\Pi^*(\alpha)$ is indeed $\frac{\sqrt{2}v(2A+t+\sqrt{2}t)}{2b(2A-2t-\sqrt{2}t)k}$. Thus,

$$\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial A} < 0, \frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial t} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial t} > 0$$

and,

$$\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial t} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial t} + \frac{\partial A}{\partial t}\left(\frac{\partial\Delta\Pi^*(\overline{\alpha}_B^*)}{\partial A} - \frac{\partial\Delta\Pi^*(\underline{\alpha}_B^*))}{\partial A}\right) > 0.$$