AR Theory from 2019 Final Coding Project

Suppose that we have the following AR(1) model:

$$y_t = \phi y_{t-1} + u_t$$

Assume that $\phi \in (0,1)$. Please derive the theoretical correlation between y_t and its second lag, y_{t-2}. Show your work!! (10 points)

ANSWER:

Given that we have an AR1 process with $\phi \in (0,1)$, it is fine to invoke that $\mathbb{E}[y_t] = 0$ (didn't invoke this in the question, but this is ok). Also, assuming the process is stationary, $Var\left(y_t\right) = Var\left(y_{t-1}\right)$.

Next, we solve for covariance between y_t and y_{t-1}, which is written as $\mathbb{E}\left[y_t y_{t-1}\right]$ (5 points for this part). Expanding for y_t, we have:

$$\begin{aligned}
\mathbb{E}\left[y_t y_{t-2}\right] &= \mathbb{E}\left[\left(\phi y_{t-1} + u_t\right) y_{t-2}\right] \\
&= \mathbb{E}\left[\left(\phi\left(\phi y_{t-2} + u_{t-1}\right) + u_t\right) y_{t-2}\right] \\
&= \mathbb{E}\left[\left(\phi\phi y_{t-2} + \phi u_{t-1} + u_t\right) y_{t-2}\right] \\
&= \mathbb{E}\left[\phi^2 y_{t-2}^2 + \phi u_{t-1} y_{t-2} + u_t y_{t-2}\right] \\
&= \phi^2 Var\left[y_{t-2}\right] + \mathbb{E}\left[\phi u_{t-1} y_{t-2}\right] + +\mathbb{E}\left[u_t y_{t-2}\right]
\end{aligned}$$

Since future shocks do not determine past values, $\mathbb{E}\left[\phi u_{t-1} y_{t-2}\right]$ and $\mathbb{E}\left[u_t y_{t-2}\right] = 0$ (this part, through involving stationarity, is 3 points). Hence:

$$\mathbb{E}\left[y_t y_{t-2}\right] = \phi^2 Var\left[y_{t-2}\right]$$

And since the process is stationary,

$$\mathbb{E}\left[y_t y_{t-2}\right] = \phi^2 Var\left[y_{t-2}\right] = \phi^2 Var\left[y_t\right]$$

Correlation is defined as the ratio of the covariance to the standard deviations of the two variables. Given that we have a stationary process, this simplifies to our answer (2 points for setting up and dividing):

$$ACF(Y_t, Y_{t-1}) = \frac{\mathbb{E}\left[y_t y_{t-2}\right]}{\sqrt{Var(y_t)}\sqrt{Var(y_{t-2})}} = \frac{\mathbb{E}\left[y_t y_{t-2}\right]}{\sqrt{Var(y_t)}\sqrt{Var(y_t)}} = \frac{\phi Var\left[y_t\right]}{Var\left[y_t\right]} = \phi^2$$