Econ 113

Lecture Module 2

Contents

Definitions

Experiment: Any manipulation of the world or observation of the world.

- Flipping a coin
- Randomized Drug trials

Sample Point: An outcome of the experiment

- One sample point must occur

  1. Heads or tails
  2. Drug success or failure

- Sample points are *mutually exclusive*

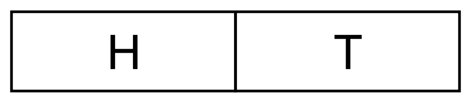
Sample space: Space of all possible outcomes

How do we represent the sample space?

Coin flip:

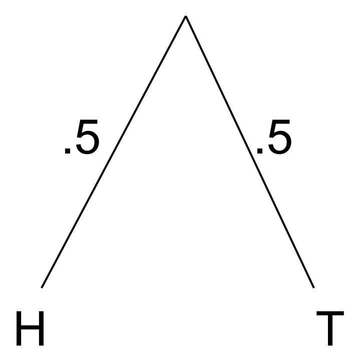
- S={H,T}
- Fair Coin

Venn diagram:



- The entire box is the space of all possible outcomes
- The space in “H” is the probability of H occurring.
- The space in “T” is the probability of T occurring.

Tree diagram:



Event: A collection of sample points

An event either happens or does not.

- One sample point is an event.
- A group of sample points is an event.
- The *complement* of an event is a way to represent the event not occurring.
- The complement of event A is denoted A^C
- Suppose that event A is heads
  - A^C is tails.

- **How do we represent the event space?**
- **New example: Flip a coin twice.**
- Sample space?
- S={HH,HT,TH,TT}
- Suppose that event A is getting heads on *exactly* one of the two flips.
- Event space?

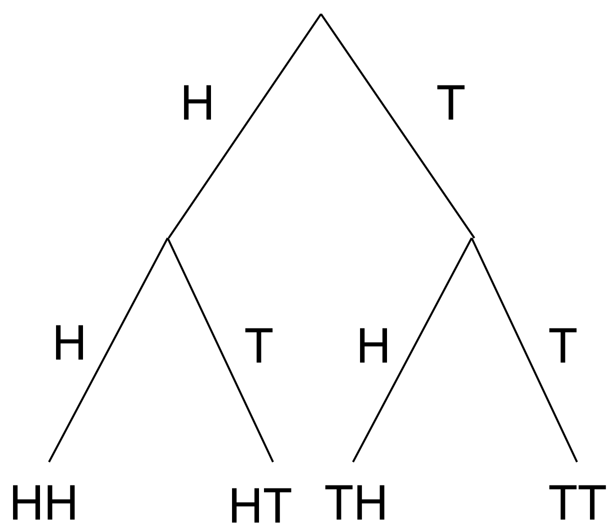
  A={HT,TH}
- What is A^c?

  A^c ={HH,TT}

Venn Diagram

HH	TH
HT	TT

Tree diagram



- **What is probability?**

  A numerical measure of how likely an event is to occur.

- **What are the key properties of probability?**

  Probabilities range from 0 to 1.

- **How do we assign probabilities?**

1. a priori classical method

   a priori is Latin for before

2. Empirical classical method

3. Subjective classical method

Probability of compound events

- **Notation**

Pr(A or B)=Pr(A U B)

Probability that either A, B, or both A and B occur.
“Union” of A and B

Pr(A and B)=Pr(A ∩ B)

Probability that both A and B occur.
“Intersection” of A and B

Example: Draw one card from a deck of cards

Event A: Drawing a face card
Event B: Drawing a spade

Pr(A U B)=Pr(Draw either a face card or spade or both)

Pr(A ∩ B)=Pr(Draw both a face card and a spade)
= Pr(Draw either a jack, queen, or king of spades)

Calculation of the union

- **General formulation**

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

- **What if events are mutually exclusive?**

Mutually exclusive means that events *<u>do not</u>* *occur together*

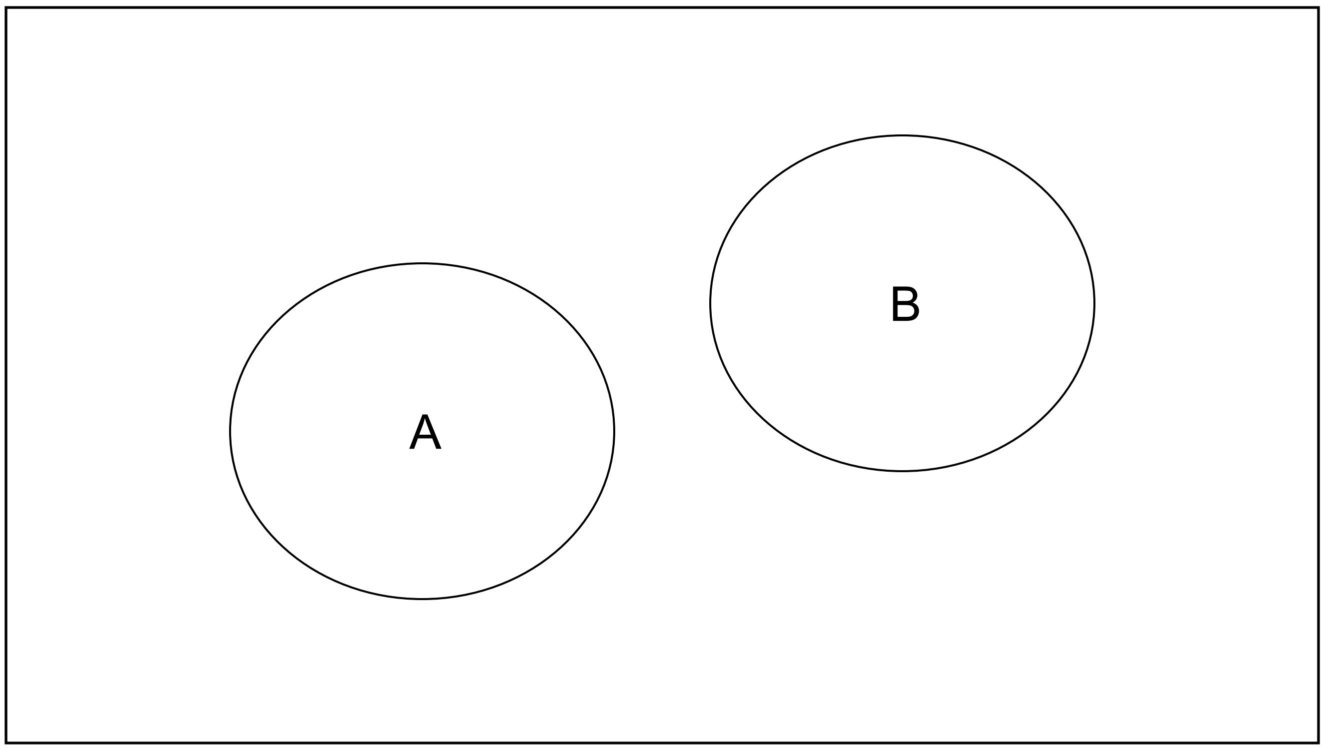
=> $Pr(A \cap B)=0$

=> $Pr(A \cup B) = Pr(A) + Pr(B)$

- **Example?**
  - Roll a die. Event A is the getting 2 or less. Event B is getting 4 or more.
  - What is $Pr(A \cup B)$?

Union of Mutually Exclusive Events

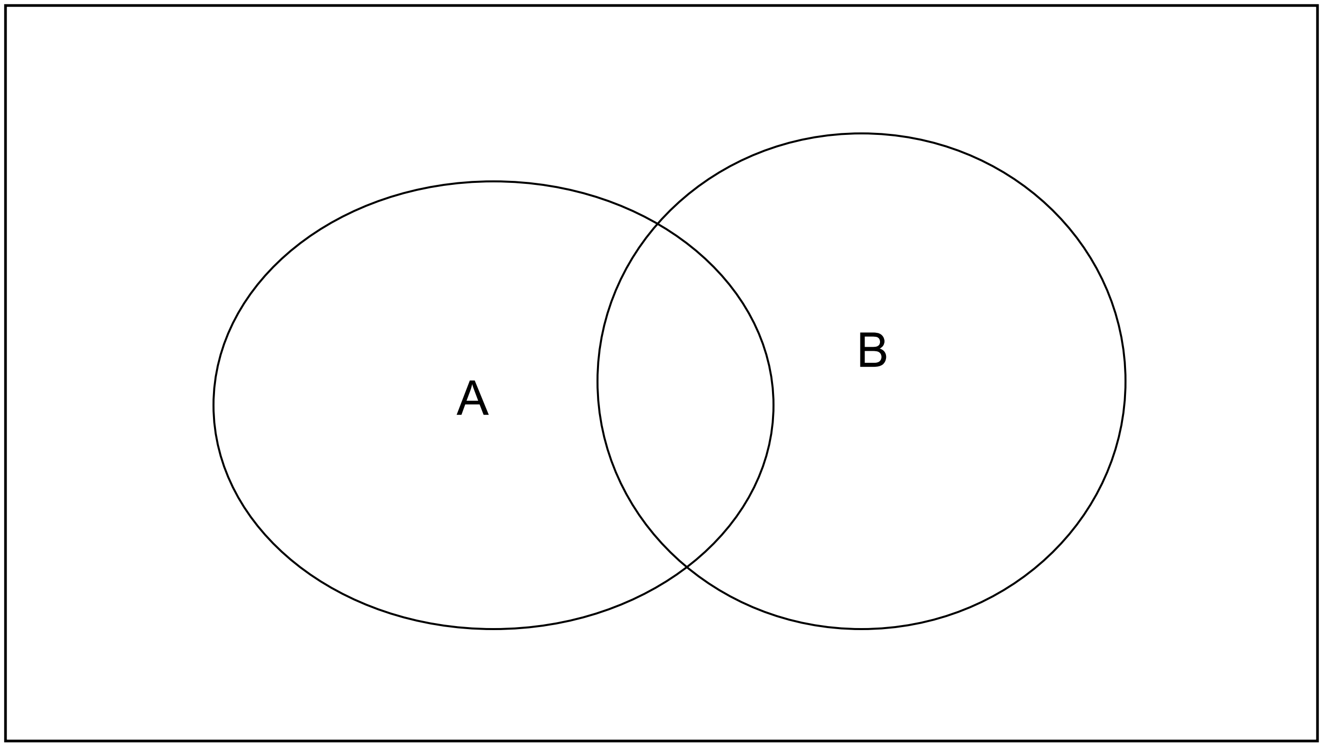
- Venn Diagram



- Events A and B do not intersect.

Union of Events

- **Events which are not mutually exclusive.**

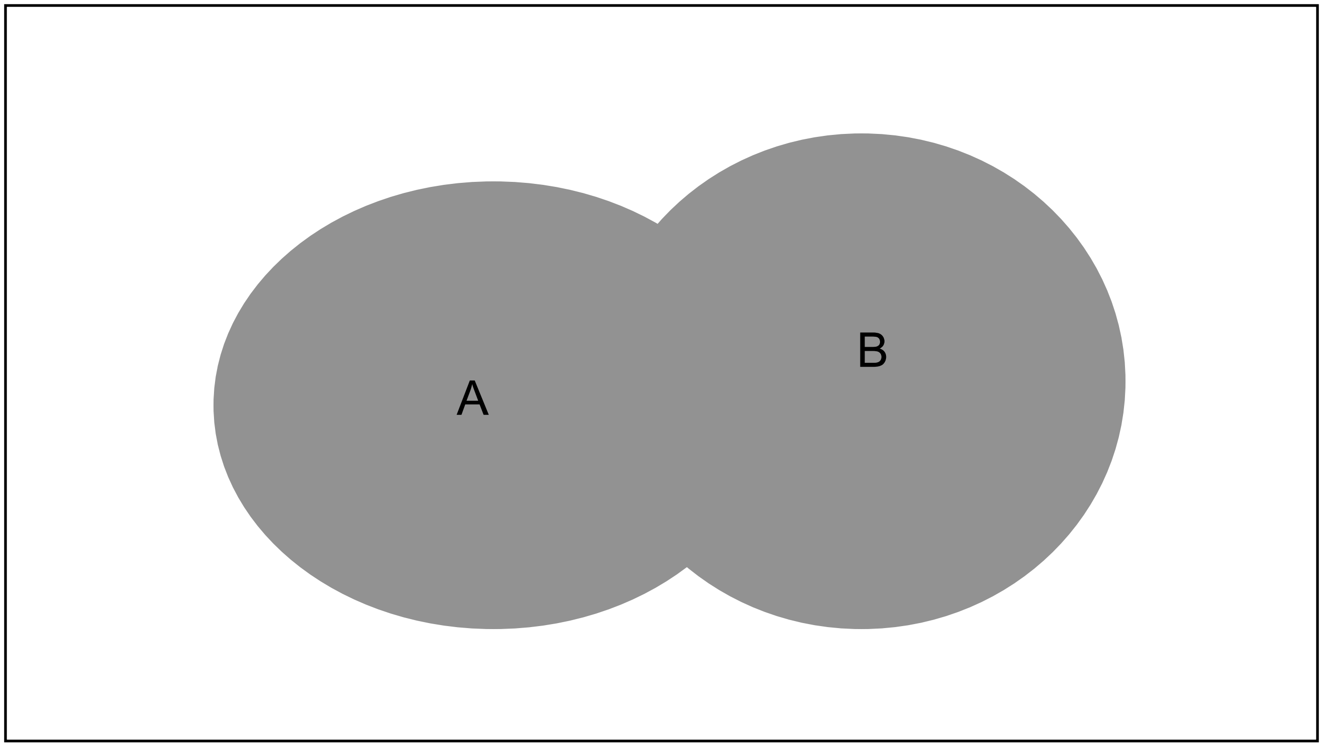


- Recall $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

  Why do we subtract $Pr(A \cap B)$?

Union of Events

- **Events which are not mutually exclusive.**



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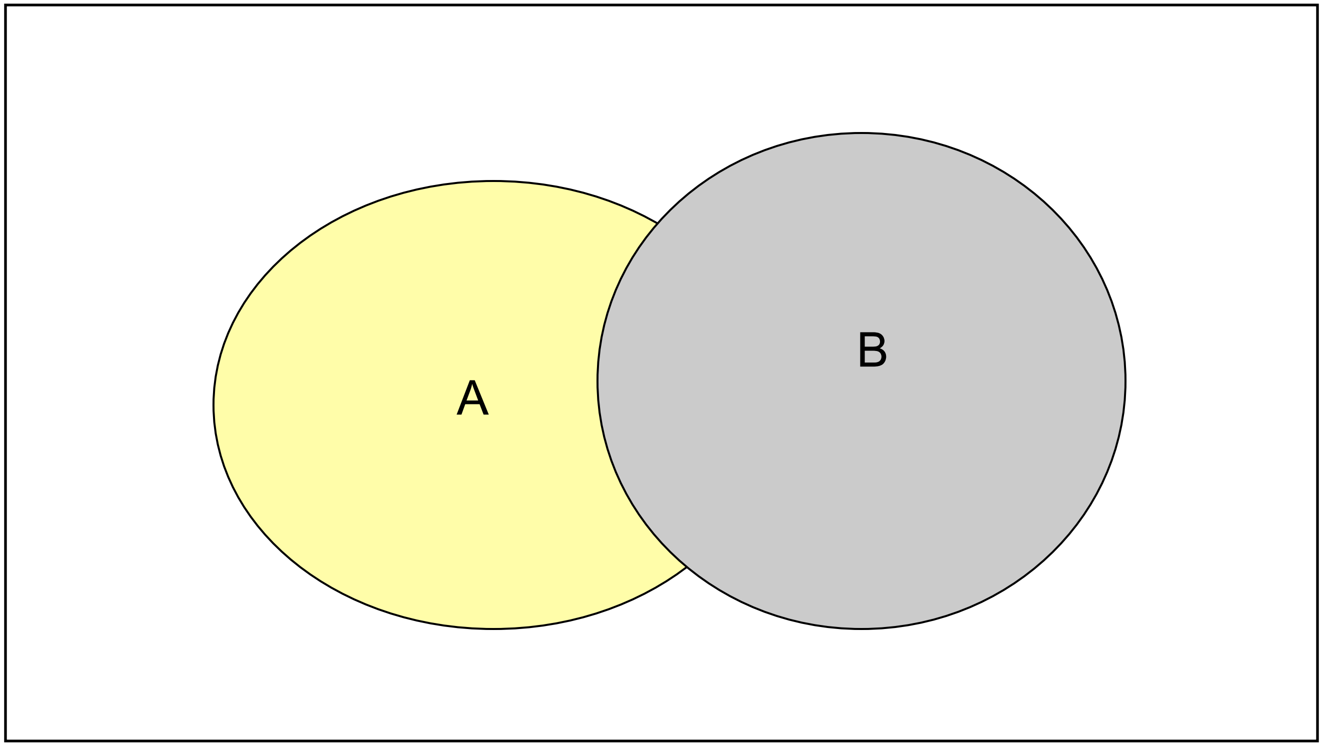

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Union

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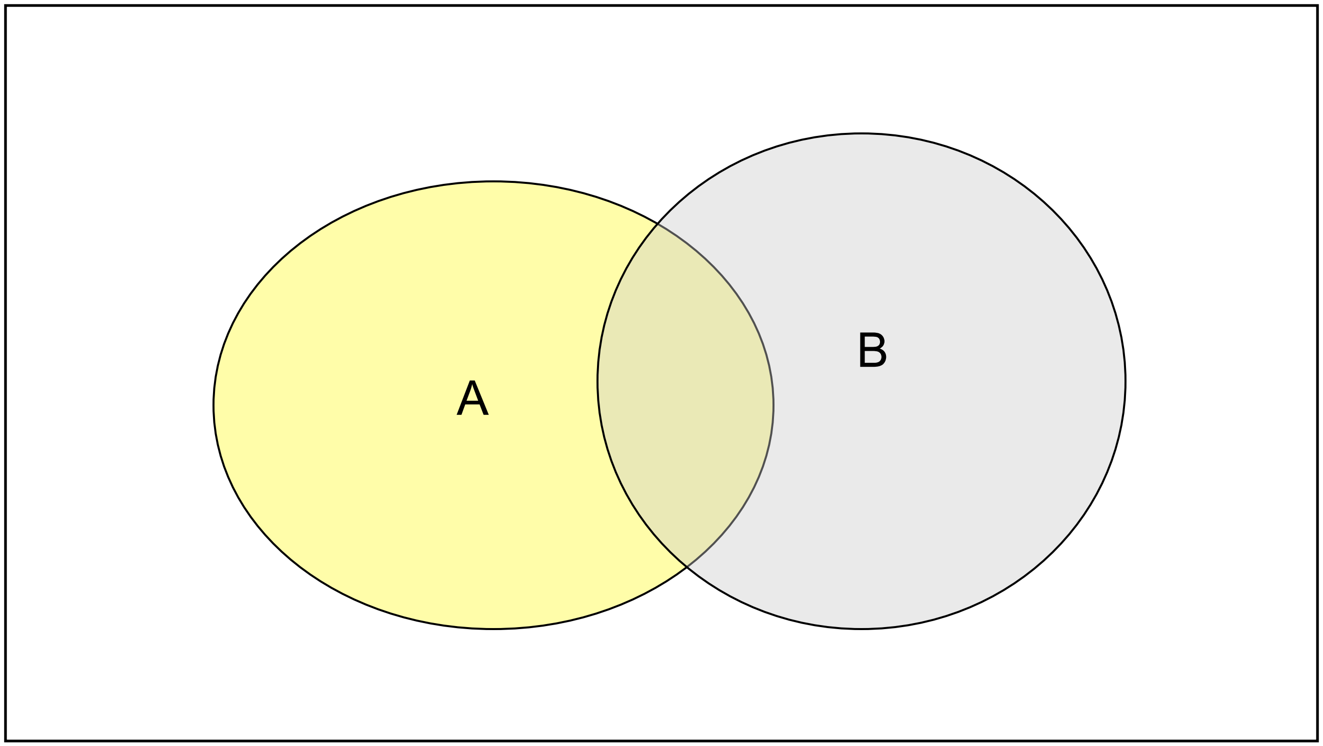


- Recall Pr(A U B) = Pr(A) + Pr(B) - Pr(A ∩ B)

  Why do we subtract Pr(A ∩ B)?

Calculation of the union

New die example

- Event A is getting an even number; Event B is getting 4 or more
- What is Pr(A)?
  Pr(A)=3/6
- What is Pr(B)?
  Pr(A)=3/6
- What is Pr(A ∩ B)?
  Pr(A ∩ B)=2/6

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 3/6 + 3/6 - 2/6$$

***If we did not subtract* 2/6, the probability of getting a 2, 4, 5 or 6 would be 1.**

Quick Review - Complements

- What is the complement of event A?
- **$Pr(A) + Pr(A^C) = 1$**
- Event A: Roll a die and get 2 or lower.

$A = \{1,2\}$

$A^C = \{3,4,5,6\}$

$$\begin{aligned} Pr(A^C) &= 1 - Pr(A) \\ &= 1 - 2/6 = 4/6 \end{aligned}$$

- We can compute $Pr(A^C)$ directly.
- Event A^C: $\{3,4,5,6\}$

=> $Pr(A^C) = 4/6$

Conditional Probability

- Probability of event A occurring given that event B has occurred $Pr(A|B)$

=> Given B, you can shrink the sample space

- **$Pr(A|B) = Pr(A \cap B)/Pr(B)$**

- Use a Venn diagram to represent conditional probability

- Example: Flip a fair coin 3 times

  Event A: Getting exactly two heads in three flips

  Event B: Getting heads on the first flip

- How do we represent this on a Venn Diagram?

Example (cont.)

- Calculate $Pr(A|B) = Pr(A \cap B)/Pr(B)$
- What is $Pr(A \cap B)$?

$\Rightarrow$ $Pr(A \cap B) = 2/8$

- What is $Pr(B)$?

$\Rightarrow$ $Pr(B) = 4/8$

$$Pr(A|B) = Pr(A \cap B)/Pr(B)$$

$$= (2/8)/(4/8)$$

$$= .5$$

Statistical Independence

- **Definition:** Events A and B are statistically independent if the occurrence of one event does not effect the probability of the other event.

- **Example:** The experiment is you draw one card from a deck.

Event A is drawing a face card from the deck in the first draw

Event B is drawing a spade from a deck of cards.

- **Multiplicative rule for independent events:** If event A and event B are statistically independent then:

  $Pr(A \text{ and } B) = Pr(A \cap B) = Pr(A)Pr(B)$

- How does this simplify the calculation of Pr(AUB)?

- How does this simplify the calculation of Pr(A|B)?

Statistical Independence

- **Example:** Flipping a rigged coin twice.

Event A is a head on the first flip.
Event B is a tail on the second flip.

$Pr(A) = 0.6$

What is $Pr(A \cap B)$?

$Pr(A \cap B) = Pr(A)P(B) = .6 * .4 = 0.24$

- **Example:** What is the probability that I get four heads in four flips?

Event A is a head on the first flip
Event B is a head on the second flip
Event C is a head on the third flip
Event D is a head on the fourth flip

What is the probability of getting 4 heads?

$Pr(A \cap B \cap C \cap D) = Pr(A)P(B)P(C)P(D) = 0.6 * 0.6 * 0.6 * 0.6 = 0.1296$

Discrete Random variables

- **Definition:** A variable that can take on only a finite number of values.

Examples?

 # heads on a series of coin flips
# of foreclosures
 # strikes per game
 # waves per hour
 # gaffes per second

- **Probability distribution of a discrete Random Variable X**

The Random variable X can take on a bunch of values x

Probability distribution is a list of all possible pairs [x,Pr(x)]

x is a value of a random variable X

Pr(x) is the probability that x occurs.

Important property: $\Sigma Pr(x) = 1$

- **Example:** Flip a fair coin 4 times. The random variable X is the number of heads.

- How many values can this take on?

  (0,1,2,3,4)

S= {HHHH,HHHT,HHTH,HTHH,THHH,HHTT,HTHT,HTTH,THTH,THHT,TTHH, HTTT,THTT,TTHT,TTTH,TTTT}

- Pr(x=0)?

  Pr(x=0) = 1/16 or [0,Pr(0)] = [0,1/16]

- Pr(x=1)?

  Pr(x=1) = 4/16 or [1,Pr(1)] = [1,4/16]

- Pr(x=2)?

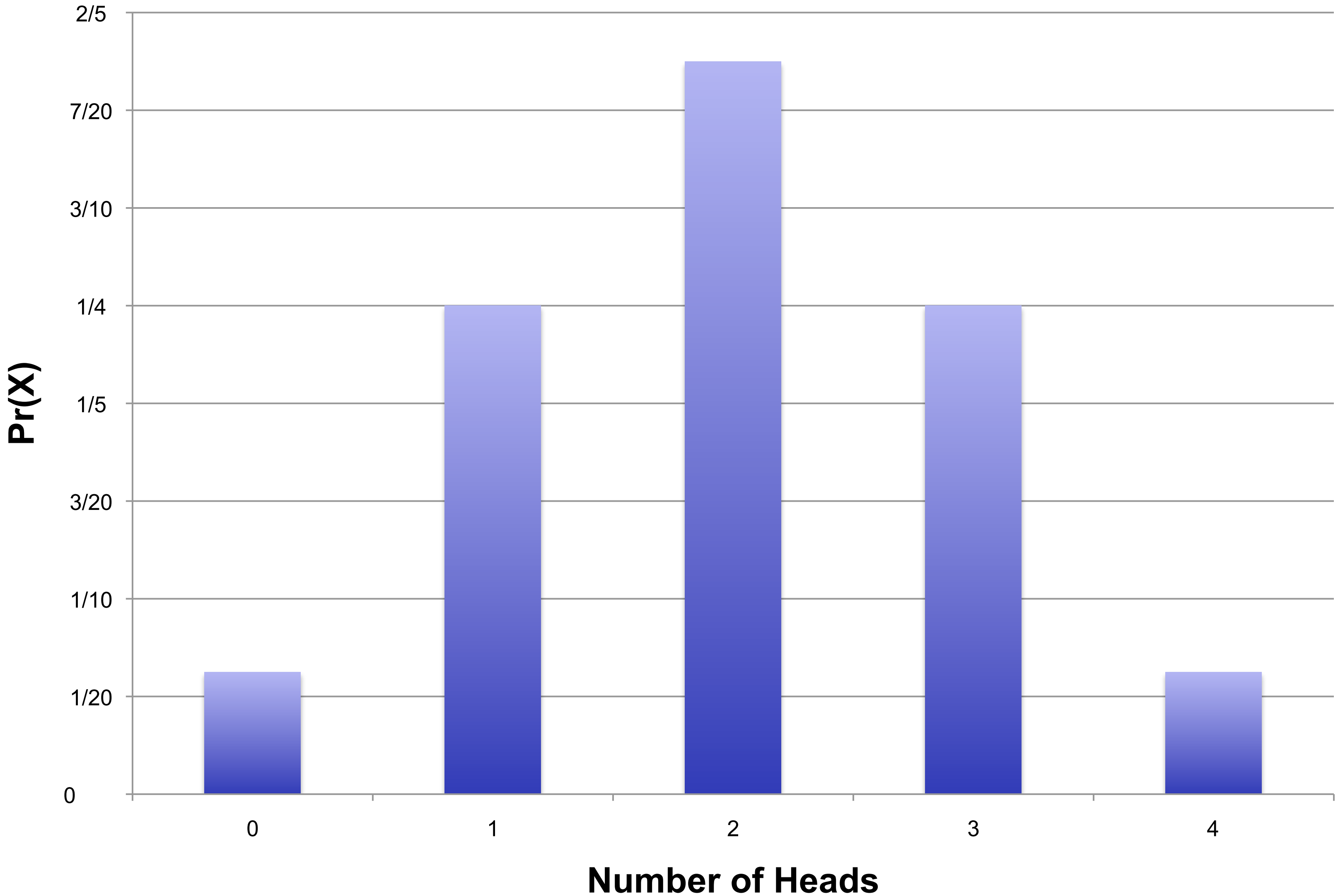
  Pr(x=2) = 6/16 or [2,Pr(2)] = [2,6/16]

- Pr(x=3)?

  Pr(x=3) = 4/16 or [3,Pr(3)] = [3,4/16]

- Pr(x=4)?

  Pr(x=4) = 1/16 or [4,Pr(4)] = [4,1/16]



2/5
7/20
3/10
1/4
1/5
3/20
1/10
1/20
0
Pr(X)
0
1
2
3
4
Number of Heads

Summary measures for Discrete Random Variables

- **Expected value:** The mean of the probability distribution

Weighted average of all possible variables

$$\mu_x = E(X) = \Sigma x Pr(x)$$

- Previous example?

- **Variance**: Weighted average of the summed deviations around the mean

$$\sigma^2 = E(x-\mu_x)^2 = \Sigma Pr(x)(x-\mu_x)^2$$

- Previous example?

Continuous Random variables

- **Definition:** A variable that can take on an infinite number of values.

Examples?

1. Consumer expenditures
2. Durations
3. Shares

- Probability distribution:

We would like to list out the probability of each event. Two problems:

1. Too many of them

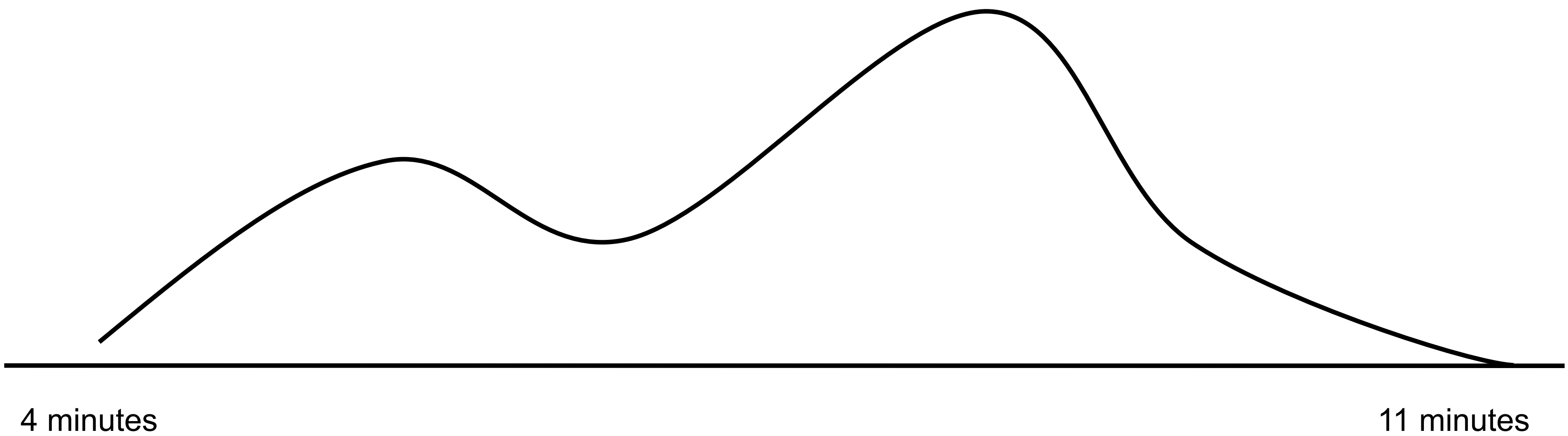
2. Chance of any given outcome happening is 0.

What is the chance that a person will run a mile in exactly 6:23.3454675678566858567896793345676654467?

Example: Distribution of one-mile times by college students

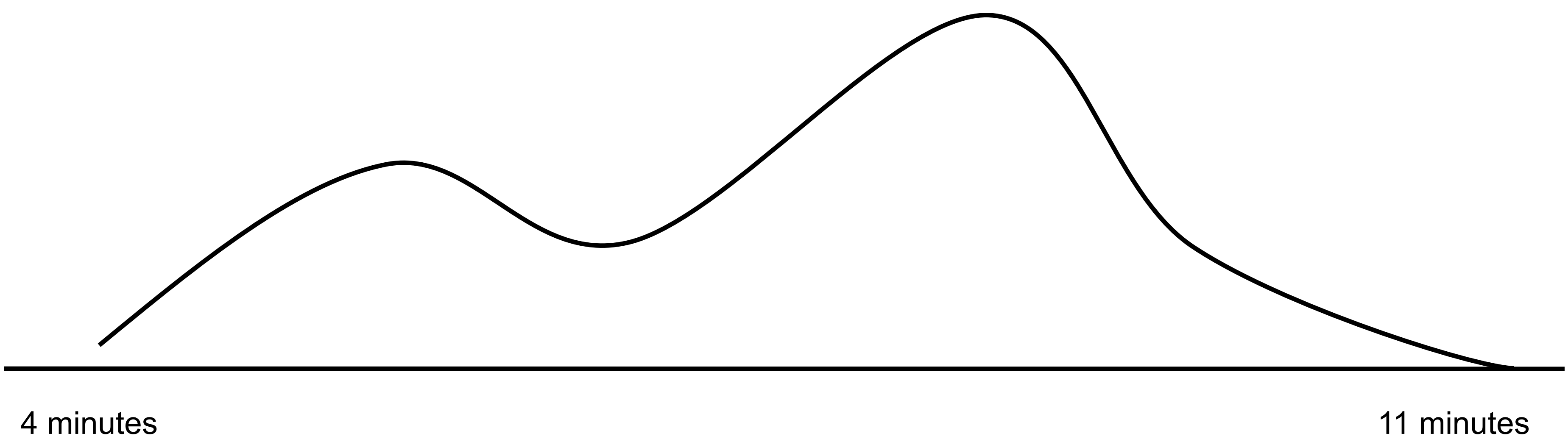
Distribution of mile times

What can we say about this distribution?



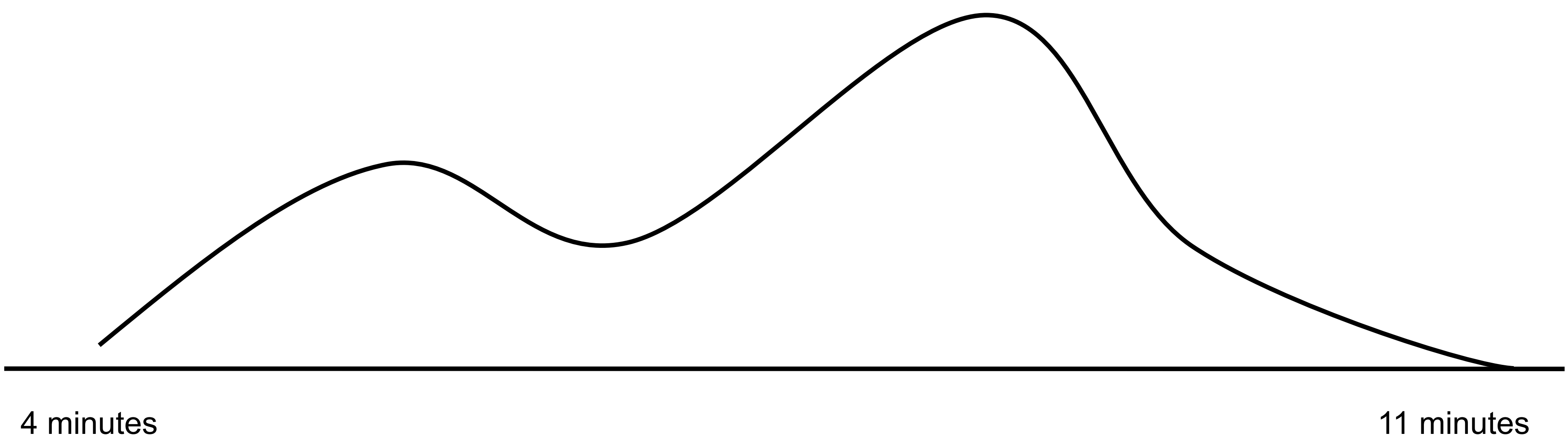
Distribution of mile times

What is the probability that a randomly selected person can run a mile in less than 6 minutes?



Distribution of mile times

What is the probability that a randomly selected person can run a mile between 8 and 9 minutes?



Uniform distribution

Any outcome is equally likely

$$f(x) = \frac{1}{d-c} \qquad \mu = \frac{c+d}{2} \qquad \sigma^2 = \frac{(d-c)^2}{12}$$

$$F(x \in S_x) = \int_{x \in S_x} \frac{1}{d-c} dx = \int_c^d \frac{1}{d-c} dx = \frac{1}{d-c} \int_c^d dx = \frac{1}{d-c}(d-c) = 1$$

- **Uniform Distribution**
- **<u>Example</u>:**
- Wave height follows a uniform distribution, [0,16]
- What is the probability that a wave is 4ft or less?

$$\Pr(a < X < b) = \int_a^b \frac{1}{d-c}\,dx$$

$$\Pr(0 < X < 4) = \int_0^4 \frac{1}{16-0}\,dx = \frac{1}{16}\int_0^4 dx = \frac{4}{16}$$

Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\mu = Mean$$

$$\sigma = SD$$

- The most important distribution in statistics
- Comes up often in nature
- Symmetric bell-shape.
- Infinite support

Small and large numbers are possible, but extremely unlikely (important!!!!)

- Distribution has mean=mode=median

Specific Distributions

<u>Normal distribution</u>

- Example: Assume heights of women are distributed normally.
- Mean = 63 and St. Dev = 3
- We seek Pr(60<Woman<70). What do we do?

$$\Pr(60 \le Woman \le 70) = \int_{60\,inches}^{70\,inches} f(x)dx$$

We want to compute the shaded area. But we can't. Why?

Standard Normal distribution

- Define a new random variable, $z=(x-\mu)/\sigma$

1. *Mean of z is zero*
2. *Standard deviation is 1*
3. *Distribution of z is normal*

- Compute z's for 70 and 60, and compute using the Z distribution (which I handed out)

$z_{60} = (60-63)/3 = -1$

$z_{70} = (70-63)/3 = 2.33$

$Pr(60<Woman<70) = Pr(z_{60}<Z<z_{70})$

$= Pr(Z<z_{70}) - Pr(Z<z_{60})$

$= Pr(Z<2.33) - Pr(Z<-1)$

$= Pr(Z<2.33) - (1-Pr(Z<1))$

$= .9901 - .1587$

$= .8314$