MATH 23 B : CHANGE OF VARIABLES THM.

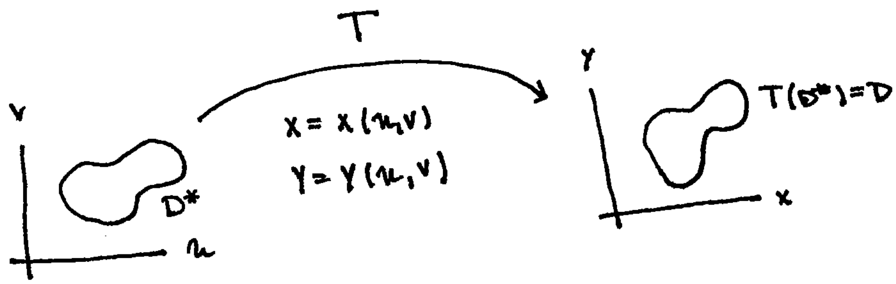
H.W. ASSIGNMENT

6.2 # 1, 2, 3, 5, 6, 9, 12, 13, 17, 19, 23, 29, 31

6.3 # 2, 7, 9, 10

DO AS IN EX. NOT LIKE SOL. GUIDE

THM: (2-DIM'L C.O.V. THM.)



D, D^* ELEM. REGIONS, $T: D^* \longrightarrow D$ C^1, ONE-TO-ONE TRANS.

IF $f: D \longrightarrow \mathbb{R}$ IS ANY INTEGRABLE FNCT. THEN

$$\iint_D f(x,y)\,dA \overset{\text{C.O.V.}}{=} \iint_{D^*} f(x(u,v), y(u,v)) \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

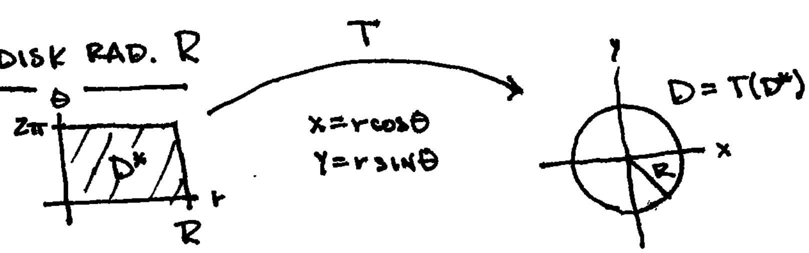
"RATHER THAN INT. OVER D WE PULL-BACK INT. TO D^* BY COMPOSING FNCT. W/ TRANS. AND ACCOUNTING FOR DISTORTION IN AREA BY JACOBIAN"

JACOBIAN DET: $T(u,v) = (x(u,v), y(u,v))$, $\dfrac{\partial(x,y)}{\partial(u,v)} = \text{DET}\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$

EX: POLAR COORD. TRANS. $T(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \text{DET}\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

AREA OF DISK RAD. R



$$A(D) = \iint_D dA \overset{\text{C.O.V.}}{=} \iint_{D^*} r\,dr\,d\theta = \int_0^{2\pi}\int_0^R r\,dr\,d\theta = \int_0^{2\pi} \frac{R^2}{2}\,d\theta = \pi R^2.$$

EXAMPLE 5 pg. 385 : GAUSSIAN INTEGRAL $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ $\leftarrow$ HAS APPLICATION IN STATISTICS

TRICK : SHOWING $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \iff \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$

X IS DUMMY VARIABLE SO WE CAN WRITE THIS

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\cdot\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \pi$$

$$\parallel$$

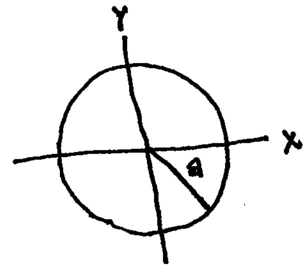
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$$

* THIS IS OUR FIRST EXAMPLE OF AN IMPROPER INT. (SEE 6.4)

WE WILL USE LIMITING PROCESS AS IN 1-VAR. CASE OF IMPROPER INTS. TO COMPUTE

WRITE $D_a = \{x^2+y^2 \le a^2\}$ $\leftarrow$ DISK RAD. $a > 0$

NOTE AS $a \longrightarrow \infty$, $D_a \longrightarrow \mathbb{R}^2$



* POLAR COORDS
$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$

$$\iint_{D_a} e^{-(x^2+y^2)}\,dA \overset{*\text{C.O.V.}}{=} \int_0^{2\pi}\int_0^{a} e^{-r^2}\cdot r\,dr\,d\theta$$

$$= 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=a} = \pi\left[-e^{-a^2}+1\right] \longrightarrow \pi \quad \text{AS } a \longrightarrow \infty$$

u-SUB.
$u = r^2$
$du = 2r\,dr$
$\int r e^{-r^2}\,dr = -\frac{1}{2}e^{-r^2}$

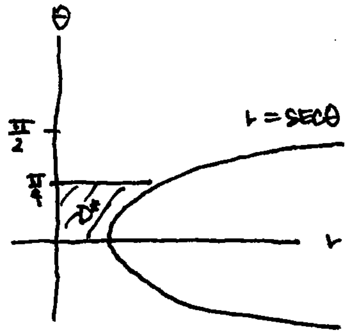
$$\therefore \lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\,dA =: \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA$$

$$\parallel$$

$$\pi \checkmark$$

31

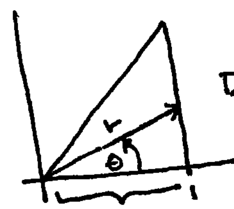
EVAL $\iint_D \sqrt{x^2+y^2}\,dA$ WHERE D IS TRIANGLE W/ VERTICES $(0,0), (1,0), (1,1)$



$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$



$D = T(D^*)$

$r\cos\theta = 1 \Rightarrow r = \sec\theta$

$D^* = \{0 \le \theta \le \frac{\pi}{4},\ 0 \le r \le \sec\theta\}$

$$\iint_D \sqrt{x^2+y^2}\,dA \overset{\text{C.O.V.}}{=} \iint_{D^*} \sqrt{r^2}\cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{4}}\int_0^{\sec\theta} r^2\,dr\,d\theta = \frac{1}{3}\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} \sec\theta\,(1+\tan^2\theta)\,d\theta \overset{*}{=} \frac{1}{3}\left(\frac{1}{2}\sec\theta\tan\theta\,\Big|_{\theta=0}^{\theta=\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec\theta\,d\theta\right)$$

$$* \int \sec\theta\cdot\tan^2\theta\,d\theta = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta\,d\theta - \int \sec\theta\tan^2\theta\,d\theta$$

INT. BY PARTS: $u = \tan\theta$ $\quad v = \sec\theta$

$du = \sec^2\theta\,d\theta \quad dv = \sec\theta\tan\theta\,d\theta$

$$\Rightarrow 2\int \sec\theta\tan^2\theta\,d\theta = \sec\theta\tan\theta - \int \sec\theta$$

$$** \int \sec\theta\,d\theta = \int \sec\theta\,\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \int \frac{1}{u}\,du = \ln u = \ln|\sec\theta+\tan\theta|$$

u-SUB.: $u = \sec\theta+\tan\theta$

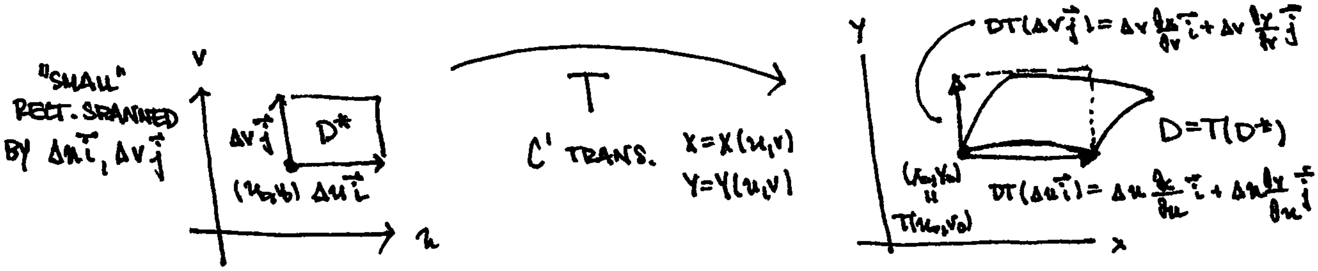
$du = \sec\theta\tan\theta + \sec^2\theta\,d\theta$

$$\Rightarrow \iint_D \sqrt{x^2+y^2}\,dA = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{\sqrt{2}} + \frac{1}{6}\ln|\sec\theta+\tan\theta|\,\Big|_{\theta=0}^{\frac{\pi}{4}} = \frac{\sqrt{2}}{6} + \frac{1}{6}\left(\ln\left|\frac{2}{\sqrt{2}}+1\right| - \ln(1+0)\right)$$

$$= \frac{\sqrt{2}}{6} + \frac{1}{6}\ln\left(\sqrt{2}+1\right).$$

#1

JUSTIFICATION FOR CHANGE OF VARS. THM. IN 2-D.: "RECALL DOUBLE INT. OVER A REGION D WAS DEFINED BY DIVIDING D INTO LITTLE RECTANGLES AND COMPUTING RIEMANN SUMS..."



IDEA: THE DERIVATIVE OF T GIVES BEST LINEAR APPROX. FOR T

$$DT = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \qquad DT(\Delta u\vec{i}) = DT\begin{pmatrix} \Delta u \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta u\frac{\partial x}{\partial u} \\ \Delta u\frac{\partial y}{\partial u} \end{pmatrix} = \Delta u\, T_u$$

$$DT(\Delta v\vec{j}) = DT\begin{pmatrix} 0 \\ \Delta v \end{pmatrix} = \begin{pmatrix} \Delta v\frac{\partial x}{\partial v} \\ \Delta v\frac{\partial y}{\partial v} \end{pmatrix} = \Delta v\, T_v$$

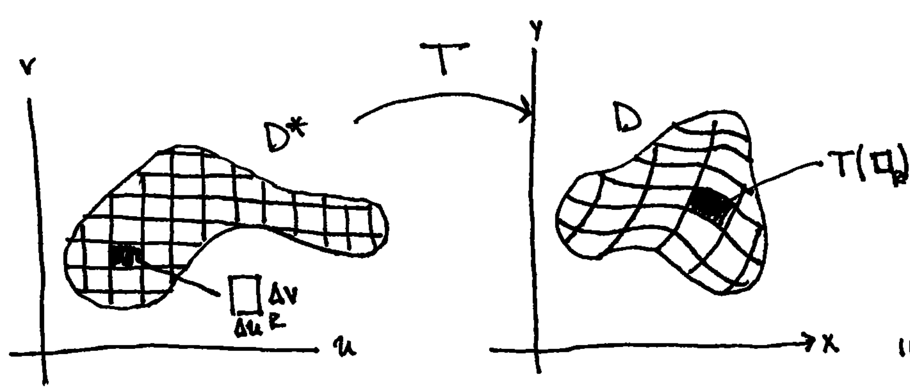
WHERE $T_u := \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}$

$T_v := \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j}$

* RECALL FROM SECTION 1.3 AREA OF PARALLELOGRAM W/ SIDES $a\vec{i}+b\vec{j}$, $c\vec{i}+d\vec{j}$ IS GIVEN BY ABS. VAL. OF DET $\left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad-bc$

* JACOBIAN MEASURES DISTORTION IN AREA BY TRANS.

$$\therefore \text{AREA}(T(D^*)) \approx \left|\det\begin{pmatrix} \Delta u\frac{\partial x}{\partial u} & \Delta u\frac{\partial y}{\partial u} \\ \Delta v\frac{\partial x}{\partial u} & \Delta v\frac{\partial y}{\partial v} \end{pmatrix}\right| = |\det(DT)|\,\Delta u\Delta v = |\text{JAC } T|\,\Delta u\Delta v$$



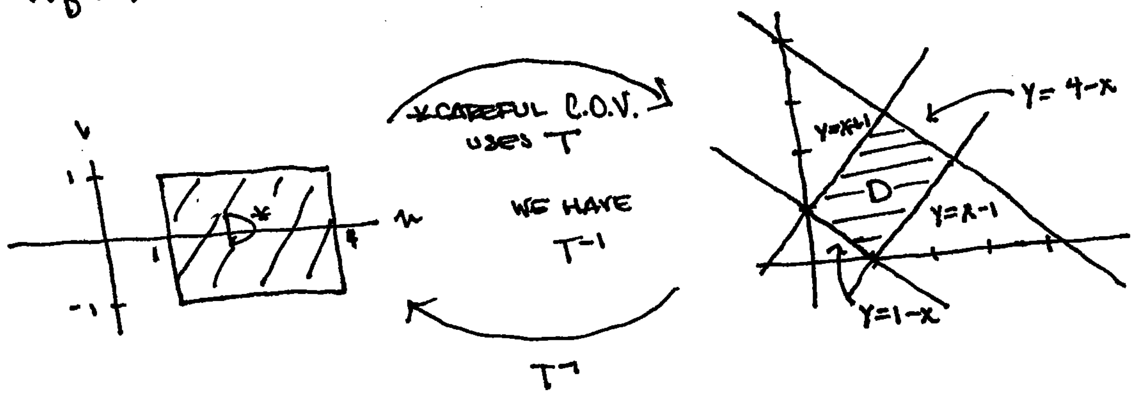
APPLYING THIS ARG. TO EACH RECT. IN PARTITION OF GEN. D^*, WE SEE

$$A(D) \approx \sum\sum \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\Delta v$$

IN THE LIMIT, THIS SUM BECOMES

$$A(D) = \iint_D dA = \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

EX: EVAL $\iint_D \left(\frac{(x-y)^2}{x+y}\right) dA$ WHERE D IS REGION BDD. BY LINES $x+y=1$, $x-y=-1$, $x+y=4$, $x-y=1$



T^{-1}

$u = x+y$
$v = x-y$

- SOMETIMES IT IS EASY TO SOLVE FOR T GIVE T^{-1} : $\boxed{\left.\begin{array}{l} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{array}\right\} T(u,v)}$

$$\text{JAC}\,T = \text{DET}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$\therefore$ BY C.O.V. $\displaystyle\iint_D \frac{(x-y)^2}{x+y}\,dA = \iint_{D^*} \frac{(v)^2}{u} |\text{JAC}\,T|\,du\,dv = \int_{-1}^{1}\int_{1}^{4} \frac{1}{2}\frac{v^2}{u}\,du\,dv$

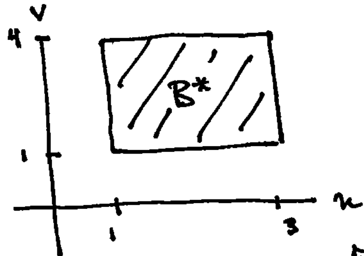
$$= \frac{1}{2}\int_{-1}^{1} v^2 \ln u \Big|_1^4 \,dv = \frac{1}{2}\int_{-1}^{1} v^2 \ln 4\,dv$$

$$= \frac{1}{2}\cdot\frac{1}{3}v^3 \ln 4 \Big|_{-1}^{1} = \frac{1}{6}\ln 4 \cdot (1-(-1)) = \frac{1}{3}\ln 4. \checkmark$$

RMK: ACTUALLY DID NOT NEED TO EXPLICITY NEED TO KNOW T OR D AS WE WILL SEE IN NEXT EXAMPLE

EX. LET B BE REGION IN FIRST QUADRANT BDD. BY CURVES $xy=1$, $x^2-y^2=1$, $xy=3$, $x^2-y^2=4$

EVAL $\iint_B x^2+y^2\,dx\,dy$ USING C.O.V. $u=xy$, $v=x^2-y^2$

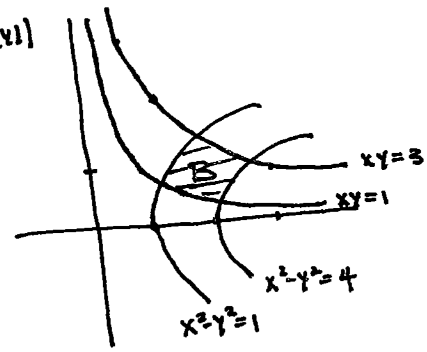


* DON'T ACTUALLY NEED TO KNOW WHAT B LOOKS LIKE BUT FOR CURIOSITY'S SAKE

$T^{-1}(x,y) = (u(x,y), v(x,y))$

$u = xy$

$v = x^2-y^2$



DON'T NEED TO SOLVE FOR T...
BY INVERSE FNCT. THM. WE CAN FIND

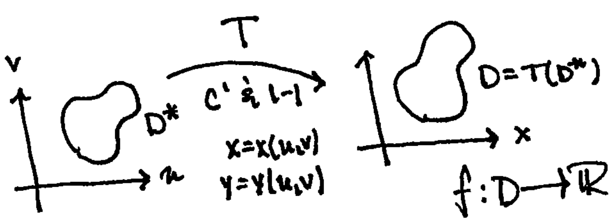
* JACT FROM T^{-1} : $\text{JACT} = \dfrac{1}{\text{JACT}^{-1}}$

$$\text{JACT}^{-1} = \det\begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix} = -2y^2-2x^2 = -2(x^2+y^2)$$

$$\Rightarrow \text{JAC}\,T = \frac{1}{-2(x^2+y^2)}$$

$$\therefore \text{ BY C.O.V. } \iint_B x^2+y^2\,dx\,dy = \iint_{B^*} x^2+y^2 \cdot |\text{JACT}|\,du\,dv = \int_1^4\int_1^3 \tfrac{1}{2}\,du\,dv = \tfrac{1}{2}\cdot 6 = 3.\checkmark$$

START WEDNESDAY LECTURE :



$f: D \to \mathbb{R}$ INT. FNCT.

THM: (2-DIM'L. C.O.V.)

$$\iint_D f(x,y)\,dA = \iint_{D^*} f(x(u,v), y(u,v))\,|\text{JACT}|\,du\,dv$$

COMPOSITION $f\circ T$

JAC. FACTOR ACCOUNTS FOR DISTORTION IN AREA

"PULL-BACK INT. FROM D TO D^* BY COMPOSING f W/ T AND ACCOUNT FOR DISTORTION IN AREA BY JAC. FAC."

C.O.V. FOR TRIPLE INTEGRALS

<u>DEF.</u>: LET $T: W^* \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ BE C^1 TRANS. DEFINED BY
$x = x(u,v,w)$
$y = y(u,v,w)$
$z = z(u,v,w)$

$$\text{JAC}\,T = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \text{DET}\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

ABS. VAL. OF JAC T IS VOL. OF PARALLELOPIPED SPANNED BY VECTORS

$$T_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$$
$$T_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$
$$T_w = \frac{\partial x}{\partial w}\vec{i} + \frac{\partial y}{\partial w}\vec{j} + \frac{\partial z}{\partial w}\vec{k}$$

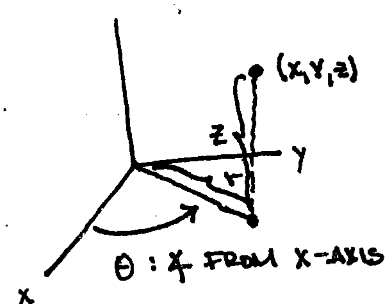
AS IN 2-DIM'L CASE, JAC (T) MEASURES HOW T DISTORTS ITS DOMAIN

<u>C.O.V. THM.</u>: LET $T: W^* \longrightarrow W$ BE C^1 TRANS., 1-1 (EXCEPT POSSIBLY ON FINITE # OF 2-DIM'L SUB-GRAPHS)

THEN $$\iiint_W f(x,y,z)\,dV = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|\text{JAC}\,T|\,du\,dv\,dw$$

2 SPECIAL COORDINATE SYSTEMS:

CYLINDRICAL COORDS.
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$



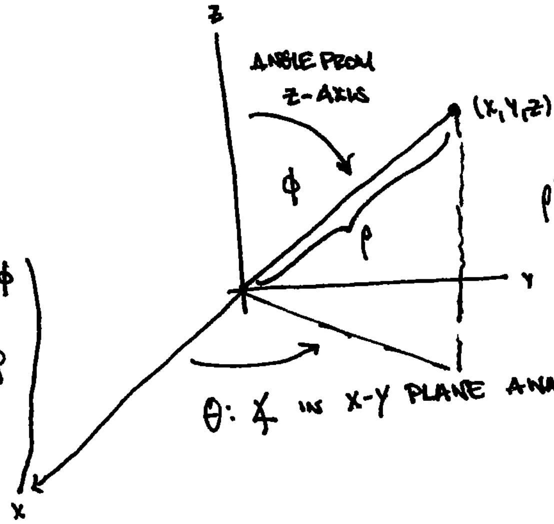
θ : ∡ FROM X-AXIS
r : RAD. IN PLANE
z : HT.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \text{DET}\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r$$

C.O.V. IN CYL. COORDS. $$\iiint_W f(x,y,z)\,dV = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\cdot r\,dr\,d\theta\,dz$$

SPHERICAL COORDS.

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$



$\rho^2 = x^2+y^2+z^2$

$\rho = \sqrt{x^2+y^2+z^2}$ ≡ LENGTH OF VEC. W/ TIP AT (x,y,z)

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \text{DET}\begin{pmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

$$= -\rho^2\cos^2\theta\sin^3\phi + (\rho\sin\theta\sin\phi)(-\rho\sin\theta\sin^2\phi - \rho\sin\theta\cos^2\phi) + \rho\cos\theta\cos\phi(\rho\cos\theta\cos\phi\sin\phi)$$

$$= -\rho^2\cos^2\theta\,\underline{\sin^3\phi} - \rho^2\sin^2\theta\sin\phi\left[\cos^2\phi+\sin^2\phi\right] + \rho^2\cos^2\theta\,\underline{\cos^2\phi}\sin\phi$$

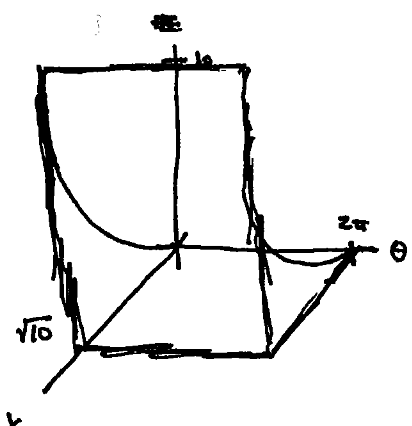
$$= -\rho^2\cos^2\theta\sin\phi(\sin^2\phi+\cos^2\phi) - \rho^2\sin^2\theta\sin\phi = -\rho^2\sin\phi$$

C.O.V. IN SPHERICAL COORDS.

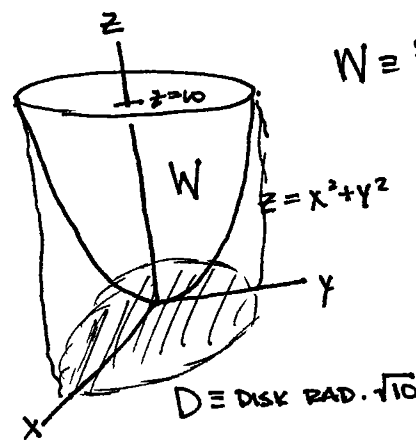
$$\iiint_W f(x,y,z)\,dV = \iiint_{W^*} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

—REVISIT H.W. PROB.—

EX: VOL. BTWN. GRAPH $z = x^2+y^2$ AND PLANE $z = 10$



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$x^2+y^2 = r^2$



W ≡ SOLID BTWN. GRAPH & PLANE

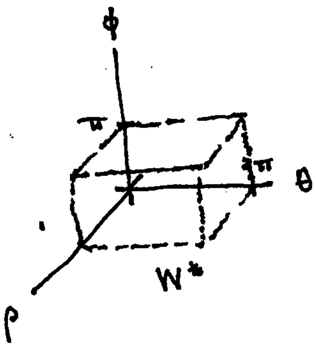
$$W^* = \{0 \le r \le \sqrt{10},\ 0 \le \theta \le 2\pi,\ r^2 \le z \le 10\}$$

$$W = \{-\sqrt{10} \le x \le \sqrt{10},\ -\sqrt{10-x^2} \le y \le \sqrt{10-x^2},\ x^2+y^2 \le z \le 10\}$$

$$\text{VOL}(W) = \iiint_W dV \overset{\text{C.O.V.}}{=} \iiint_{W^*} r\,dr\,d\theta\,dz = \int_0^{\sqrt{10}}\int_0^{2\pi}\int_{r^2}^{10} r\,dz\,d\theta\,dr = 2\pi\int_0^{\sqrt{10}} 10r - r^3\,dr$$

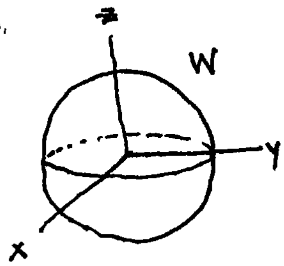
$$= 2\pi\left[5r^2 - \tfrac{1}{4}r^4\right] = 2\pi\left[5\cdot 10 - 25\right] = 50\pi \checkmark$$

EX. 6 pg. 389: $\iiint_W e^{(x^2+y^2+z^2)^{3/2}}\,dV$ W = UNIT BALL IN $\mathbb{R}^3$



$\xrightarrow{T}$

$x = \rho\cos\theta\sin\phi$
$y = \rho\sin\theta\sin\phi$
$z = \rho\cos\phi$



BY C.O.V. $\iiint_W e^{(x^2+y^2+z^2)^{3/2}}\,dV = \iiint_{W^*} e^{(\rho^2)^{3/2}}\cdot \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

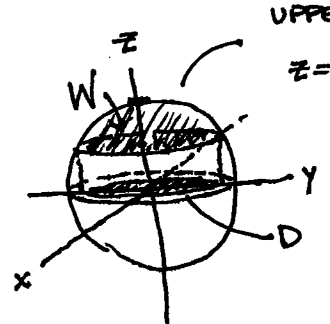
$$= \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{\rho^3}\cdot\rho^2\cdot\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{2\pi} e^{\rho^3}\Big|_0^1\cdot\sin\phi\,d\theta\,d\phi = \frac{2\pi}{3}\int_0^{\pi}(e-1)\sin\phi\,d\phi$$

$$= -\frac{2\pi}{3}(e-1)\cos\phi\Big|_0^{\pi} = -\frac{2\pi}{3}(e-1)\,[-1-1] = \frac{4\pi}{3}(e-1)\ \checkmark$$

EX: SET UP TRIPLE INTS. IN SPECIFIED COORD. SYSTEM THAT WOULD GIVE VOL. INSIDE SPHERE $x^2+y^2+z^2=4$ AND ABOVE PLANE $z=1$

UPPER HEMI-SPHERE $z=\sqrt{4-x^2-y^2}$



RECTANGULAR COORDINATES:

$$\text{VOL}(W) = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$$

$W = \{(x,y)\in D \mid 1 \le z \le \sqrt{4-x^2-y^2}\}$

INTERSECTION: $z=1 \Rightarrow 4 = x^2+y^2+z^2 = x^2+y^2+1$
$\Rightarrow 3 = x^2+y^2$

D = PROJ. OF W INTO X-Y PLANE = DISK RAD. $\sqrt{3}$
$= \{-\sqrt{3}\le x\le\sqrt{3},\ -\sqrt{3-x^2}\le y\le\sqrt{3-x^2}\}$

CYL. COORDS.: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$D = $ DISK RAD. $\sqrt{3}$ $= \{0 \le r \le \sqrt{3},\ 0 \le \theta \le 2\pi\}$

$\therefore\ W^* = \{0 \le r \le \sqrt{3},\ 0 \le \theta \le 2\pi,\ 1 \le z \le \sqrt{4-r^2}\}$

Z LIMITS: CARTESIAN

$1 \le z \le \sqrt{4-x^2-y^2}$

CYL.: $1 \le z \le \sqrt{4-r^2}$

SINCE $x^2+y^2=r^2$

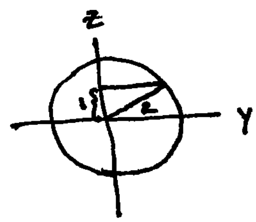
BY C.O.V. THM. JAC. FACTOR

$$\text{VOL}(W) = \iiint_W dV \overset{\text{CYL. COORDS.}}{=} \iiint_{W^*} r\, dz\, d\theta\, dr$$

$$= \int_0^{\sqrt{3}} \int_0^{2\pi} \int_1^{\sqrt{4-r^2}} r\, dz\, d\theta\, dr$$

SPHERICAL COORDS: $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$

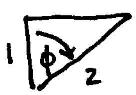
θ LIMITS: $0 \le \theta \le 2\pi$



ϕ LIMITS: TAKE Y-Z SLICE

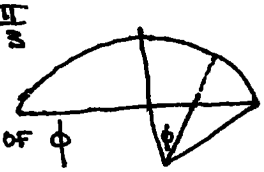
ϕ ∡ FROM Z-AXIS

STARTS AT $\phi = 0$



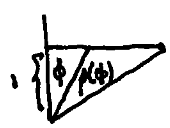
$\cos\phi = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$

SO $0 \le \phi \le \frac{\pi}{3}$



ρ LIMITS: IDEA: FIND ρ AS FNCT. OF ϕ

ρ STARTS AT $\rho(\phi)$ AND GOES TO 2



$\cos\phi = \frac{1}{\rho(\phi)} \Rightarrow \rho(\phi) = \frac{1}{\cos\phi} = \sec\phi$

$W^* = \{0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{3},\ \sec\phi \le \rho \le 2\}$

$$\therefore\ \text{VOL}(W) = \iiint_W dV = \iiint_{W^*} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\phi}^{2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

EX: USE $\int$ OF PREV. TRIPLE INTS. TO COMPUTE VOL. W

BEST CHOICE : CYL. COORDS.
(EASIEST)

$$\text{Vol}(W) = \int_0^{\sqrt{3}}\int_0^{2\pi}\int_1^{\sqrt{4-r^2}} r\,dz\,d\theta\,dr = \int_0^{\sqrt{3}}\int_0^{2\pi} r\sqrt{4-r^2} - r\,d\theta\,dr$$

$$= 2\pi\int_0^{\sqrt{3}} r\sqrt{4-r^2} - r\,dr \overset{*}{=} 2\pi\left[-\frac{1}{3}(4-r^2)^{\frac{3}{2}} - \frac{1}{2}r^2\right]_{r=0}^{r=\sqrt{3}}$$

$$= 2\pi\left[\left(-\frac{1}{3}(4-3)^{\frac{3}{2}} - \left(-\frac{1}{3}(4)^{\frac{3}{2}}\right)\right) - \frac{1}{2}3 - \left(-\frac{1}{2}\cdot 0\right)\right]$$

$$= 2\pi\left[-\frac{1}{3} + \frac{8}{3} - \frac{3}{2}\right]$$

$$= 2\pi\left[\frac{7}{3} - \frac{3}{2}\right] = 2\pi\left[\frac{14}{6} - \frac{9}{6}\right]$$

$$= 2\pi\cdot\frac{5}{6} = \frac{5\pi}{3} \checkmark.$$

* u-SUB.: $u = 4-r^2$
$du = -2r\,dr$

$$\int r\sqrt{4-r^2} = -\frac{1}{2}\int\sqrt{u}\,du$$
$$= -\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} = -\frac{1}{3}(4-r^2)^{3/2}$$

APPLICATIONS : 6.3