Summary of 6.2 & 6.3

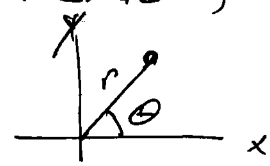
Change of variables in 2-dim

Let $T: D^* \to D$ be 1-1 & onto. Then

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Special case: $T(r,\theta) = (r\cos\theta, r\sin\theta)$ (Polar)

Jacobian: $\frac{\partial(x,y)}{\partial(r,\theta)} = r$



Change of var in 3-dim

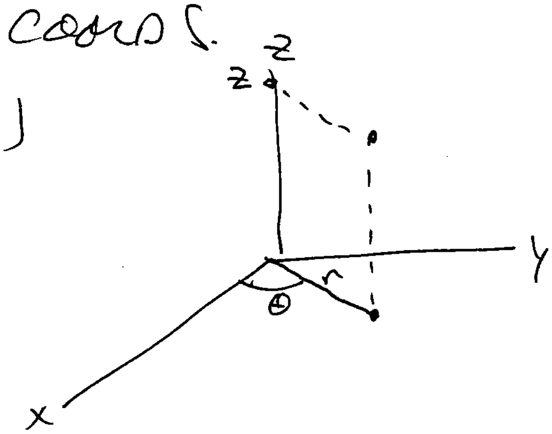
Let $T: W^* \to W$ be 1-1 & onto. Then

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw.$$

Special case: Cylindrical coords.

$T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$

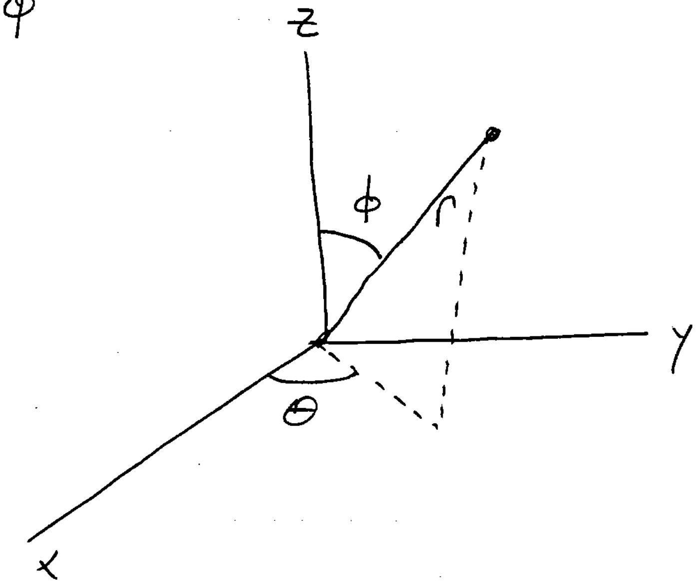
Jacobian: r



SPECIAL CASE: SPHERICAL COORDS.

$$T(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)$$

JACOBIAN: $\rho^2\sin\phi$



SUPPOSE W IS A SOLID REGION WITH MASS DENSITY FUNCTION $\delta(x,y,z)$.

$$\text{total mass} = \iiint_W \delta(x,y,z)\,dx\,dy\,dz$$

CENTER OF MASS

$$\bar{x} = \frac{\iiint_W x\,\delta(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

$$\bar{y} = \frac{\iiint_W y\,\delta(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

$$\bar{z} = \frac{\iiint_W z\,\delta(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

MOMENTS OF INERTIA:

$$I_x = \iiint_W (y^2+z^2)\,\delta(x,y,z)\,dx\,dy\,dz$$

$$I_y = \iiint_W (x^2+z^2)\,\delta(x,y,z)\,dx\,dy\,dz$$

$$I_z = \iiint_W (x^2+y^2)\,\delta(x,y,z)\,dx\,dy\,dz$$

& ANALOGOUS FORMULAS IN 2-dim.

6.4 IMPROPER INTEGRALS

TWO TYPES OF IMPROPER INTEGRALS

(1) UNBOUNDED REGIONS

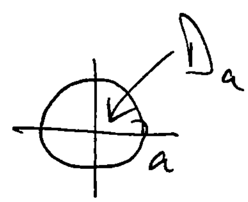
(2) INTEGRAND 'BLOWS UP' AT SOME POINT OR POINTS (OFTEN ON ~~THE~~ THE BOUNDARY.)

(1) EX 5 FROM 6.2 (GAUSSIAN INTEGRAL)

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

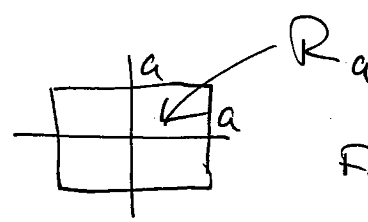
DO $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy$ IN TWO WAYS

- $\displaystyle\lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy$



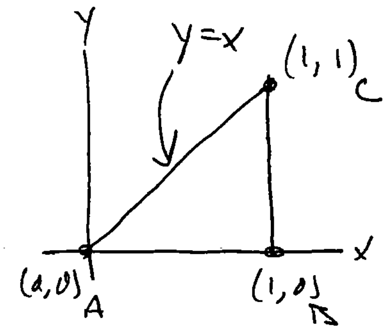
SWITCH TO POLAR

- $\displaystyle\lim_{a\to\infty} \iint_{R_a} e^{-(x^2+y^2)}\,dx\,dy$



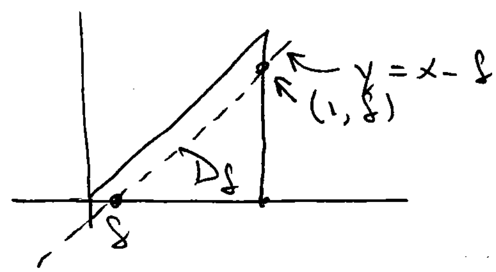
FACTOR.

(2) Ex. $\iint_D \frac{1}{\sqrt{x-y}}\,dx\,dy \quad D:\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \\ y \le x \end{cases}$



INTEGRAND BLOWS UP ON $y=x$. COULD DO FOLLOWING FOR $\delta>0$.

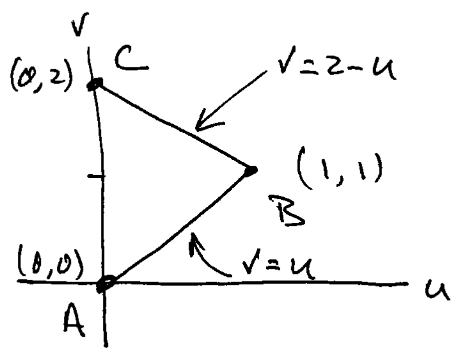
$$D_\delta:\begin{cases} \delta \le x \le 1 \\ 0 \le y \le 1-\delta \\ y \le x-\delta \end{cases}$$



TAKE $\displaystyle\lim_{\delta\to 0}\iint_{D_\delta}\frac{1}{\sqrt{x-y}}\,dx\,dy$.

INSTEAD WE CAN FIRST TRANSFORM, THEN TAKE limit.

$$T:\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(v-u) \end{cases} \quad \text{so} \quad \begin{cases} u = x-y \\ v = x+y \end{cases}$$

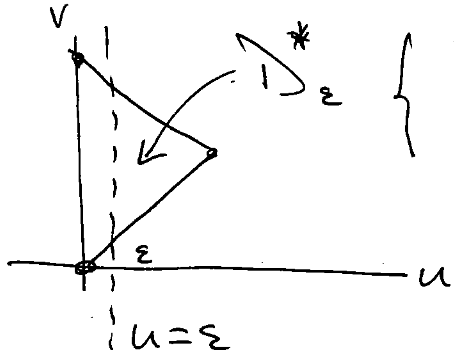


$$D^*:\begin{cases} 0 \le u \le 1 \\ u \le v \le 2-u \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\therefore \iint_D \frac{1}{\sqrt{x-y}}\, dx\,dy = \iint_{D^*} \frac{1}{\sqrt{u}} \cdot \frac{1}{2}\, du\,dv$$

NOW INTEGRAND BLOWS UP ON LINE $u=0$.



$D^*_\varepsilon \begin{cases} \varepsilon \le u \le 1 \\ u \le v \le 2-u \end{cases}$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} \int_\varepsilon^1 \int_u^{2-u} u^{-\frac{1}{2}}\, dv\,du = \lim_{\varepsilon \to 0} \frac{1}{2} \int_\varepsilon^1 u^{-\frac{1}{2}}\, v \Big|_u^{2-u}\, du$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} \int_\varepsilon^1 u^{-\frac{1}{2}} (2-u-u)\, du = \lim_{\varepsilon \to 0} \cancel{\tfrac{1}{2}} \int_\varepsilon^1 u^{-\frac{1}{2}} (1-u) \cdot \cancel{2}\, du$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^1 \left(u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du = \lim_{\varepsilon \to 0} \left(2u^{\frac{1}{2}} - \frac{2}{3} u^{3/2}\right)\Big|_\varepsilon^1$$

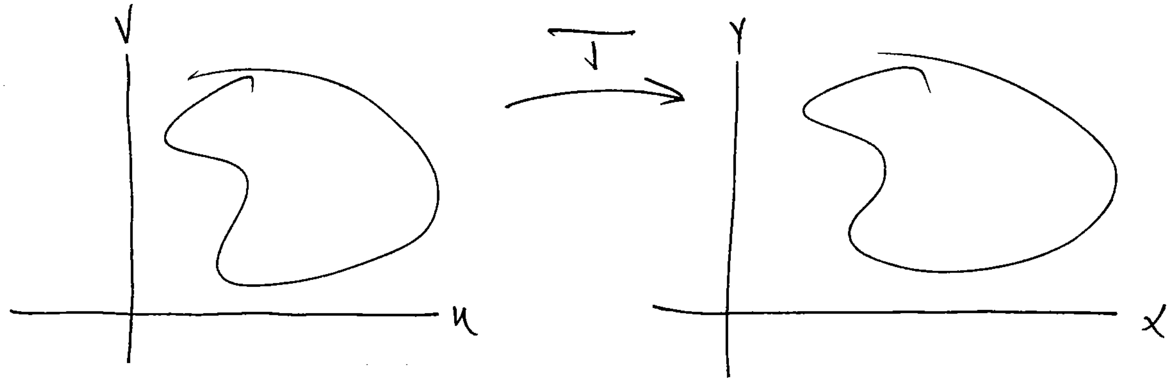
$$= \lim_{\varepsilon \to 0} \left(\left(2 - \frac{2}{3}\right) - \left(2\varepsilon^{1/2} - \frac{2}{3}\varepsilon^{3/2}\right) \right) = 2 - \frac{2}{3} = \boxed{\frac{4}{3}}.$$

(6.4) HW: 1, 3, 4, 5, 8

BACK TO AN EX FROM 6.2!

$T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(u,v) = (u^2 - v^2, 2uv)$

THIS MAPPING COMES UP IN SEVERAL (#31, #6) HW PROBLEMS. IT HAS AN INTERESTING GEOMETRICAL INTERPRETATION



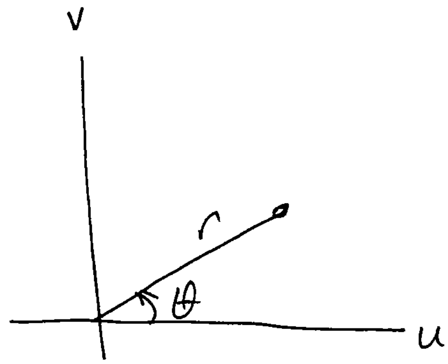
$$T: \begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$

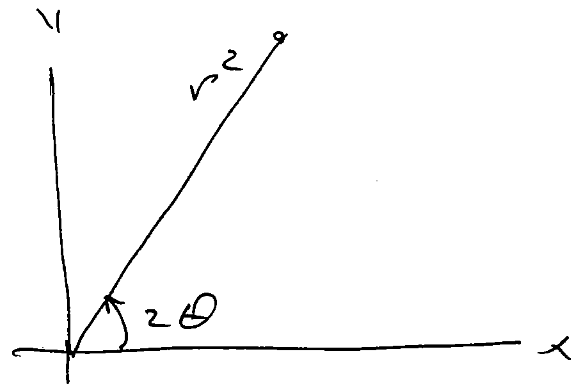
TO SEE WHAT T DOES, LOOK AT UV PLANE IN POLAR COORDINATES:

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$$

$$\therefore \quad x = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos(2\theta)$$
$$y = 2r\cos\theta\, r\sin\theta = r^2(2\sin\theta\cos\theta) = r^2\sin(2\theta).$$

THUS T TAKES THE POINT IN uv PLANE WITH POLAR COORDINATES (r, θ) TO THE POINT IN xy PLANE WITH POLAR COORDINATES $(r^2, 2\theta)$.





i.e. T DOUBLES ANGLES & SQUARES DISTANCE.

IN THESE COORDINATES IT'S EASY TO SEE THAT T IS ONE-TO-ONE (#31 a) SINCE

$$\left\{\begin{array}{l} r^2 = r'^2 \ \& \\ 2\theta = 2\theta' \end{array}\right. \Longrightarrow \left\{\begin{array}{l} r = r' \\ \theta = \theta' \end{array}\right.$$