(8.4) Gauss' Theorem

Recall that another way to state Green's theorem in $\mathbb{R}^2$ was as

$$\int_{\partial D} \vec{F}\cdot\vec{N}\, ds = \iint_D \mathrm{div}(\vec{F})\, dA$$

where $\vec{N}$ is the outward pointing unit normal to ∂D.

This generalizes to $\mathbb{R}^3$ as follows

Theorem (Gauss' Divergence Theorem)

Let W be a region in $\mathbb{R}^3$ and ∂W be the oriented surface (with outward normal) which bounds W. Let $\vec{F}$ be a C^1 vector field on W. Then

$$\iiint_W \mathrm{div}(\vec{F})\, dV = \iint_{\partial W} \vec{F}\cdot d\vec{S}$$

i.e.

$$\iiint_W (\nabla\cdot\vec{F})\, dV = \iint_{\partial W} (\vec{F}\cdot\vec{N})\, dS$$

where $\vec{N}$ is the outward unit normal to ∂W.

THE PROOF IS SIMILAR TO THAT OF GREEN'S THEOREM AND WE OMIT IT. (SEE P. 564-66.)

Ex. $\vec{F} = (2x, y^2, z^2)$, $S: x^2+y^2+z^2=1$.

FIND $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = \iint_S (\vec{F}\cdot\vec{N})\, dS$.

HARD WAY:

$\vec{N} = (x, y, z)$ SO $\vec{F}\cdot\vec{N} = 2x^2 + y^3 + z^3$

$$\begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases} \qquad |T_\phi \times T_\theta| = \cdots = \sin\phi$$

SO

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S (2x^2+y^3+z^3)\, dS$$

$$= \int_0^{2\pi}\int_0^{\pi} (2\sin^2\phi\cos^2\theta + \sin^3\phi\sin^3\theta + \cos^3\phi)\sin\phi \, d\phi\, d\theta$$

$$= \cdots \text{EXERCISE} \cdots = \frac{8\pi}{3}$$

EASY WAY:

$$\operatorname{div}(\vec{F}) = 2 + 2y + 2z$$

LET W BE THE UNIT BALL WITH $\partial W = S$. THEN

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\, dV$$

$$= 2\iiint_W (1 + y + z)\, dV$$

$$= 2\iiint_W dV + 2\iiint_W y\, dV + 2\iiint_W z\, dV$$

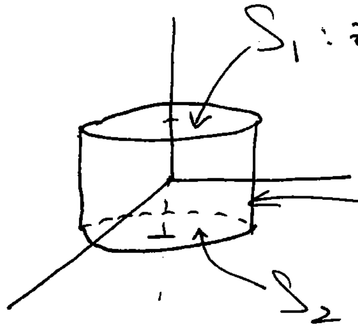
BY SYMMETRY WE SEE $\iiint_W y\, dV = \iiint_W z\, dV = 0$

THUS

$$\iint_S \vec{F}\cdot d\vec{S} = 2\,\text{Volume}(W) = 2\cdot\frac{4\pi}{3} = \boxed{\frac{8\pi}{3}}.$$

<u>EX</u> $\vec{F} = (xy^2, x^2y, y)$ AND LET S

BE SURFACE OF CYLINDER $x^2+y^2=1$ BOUNDED BY THE PLANES $z=1$ AND $z=-1$



$$S = S_1 \cup S_2 \cup S_3$$

EVALUATE $\iint_S \vec{F} \cdot d\vec{S}$.

WE HAVE

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} + \iint_{S_3} \vec{F} \cdot d\vec{S}$$

$$= \ldots \text{ EXERCISE } \ldots = \pi$$

<u>EASY WAY</u>: $\mathrm{div}(\vec{F}) = x^2+y^2$

NOTE $S = \partial W$ WHERE W IS GIVEN BY

$$W : x^2+y^2 \le 1, \quad -1 \le z \le 1.$$

So

Cylindrical Coords:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \\ dV = r\,dz\,dr\,d\theta \end{cases}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W (x^2+y^2)\,dV$$

$$= \int_0^{2\pi}\int_0^1\int_{-1}^1 r^3\,dz\,dr\,d\theta$$

$$= 2\cdot 2\pi \cdot \left.\frac{r^4}{4}\right|_0^1 = 4\pi\cdot\frac{1}{4} = \boxed{\pi}$$

<u>PHYSICAL INTERPRETATION OF DIVERGENCE</u>

REGARD $\vec{F}$ AS THE VELOCITY FIELD OF A FLUID MOVING IN SPACE. PLACE A DROP OF BLACK INK INTO THE FLUID AT A POINT P. THEN $\mathrm{div}(\vec{F})(P)$ IS THE INSTANTANEOUS RATE OF CHANGE IN INK VOLUME PER UNIT VOLUME (i.e. $\mathrm{div}(\vec{F})(P)$ IS A PERCENTAGE.)

IF $\mathrm{div}(\vec{F})(P) > 0$, P IS CALLED A <u>SOURCE</u> AND IF $\mathrm{div}(\vec{F}) < 0$, P IS CALLED A <u>SINK</u> FOR THE FLUID.

IF $\mathrm{div}(\vec{F})(P) = 0$ AT EVERY POINT P, THEN $\vec{F}$ IS SAID TO BE <u>DIVERGENCE FREE</u> AND THE FLUID IS SAID TO BE <u>INCOMPRESSIBLE</u>.

NOTE, IF $\operatorname{div}(\vec{F}) = 0$ THEN BY THE GAUSS' THM

$$\iint_S \vec{F} \cdot d\vec{S} = 0$$

FOR <u>EVERY</u> CLOSED SURFACE S. i.e. THE FLUX OF THIS 'FLUID' ACROSS ANY CLOSED SURFACE ZERO, i.e. AS MUCH FLUID FLOWS INTO S AS OUT OF S.