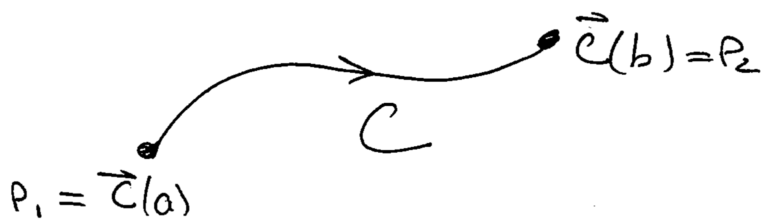
(8.3) CONSERVATIVE FIELDS

WE SAW IN 7.2 THAT IF $\vec{F} = \nabla f$ FOR SOME FUNCTION $f: \mathbb{R}^3 \to \mathbb{R}$ (OR $\mathbb{R}^2 \to \mathbb{R}$) AND $c: [a,b] \to \mathbb{R}^3$ IS A PARAMETRIZATION OF A CURVE C (ORIENTED)



THEN $\displaystyle\int_C \vec{F} \cdot d\vec{s} = f(P_2) - f(P_1)$, i.e.

THE INTEGRAL DEPENDS ONLY ON THE ENDPOINTS OF THE CURVE C. THUS ANY OTHER CURVE C_2 (ORIENTED) WOULD GIVE THE SAME RESULT

$$\int_C \vec{F} \cdot d\vec{s} = f(P_2) - f(P_1) = \int_{C_2} \vec{F} \cdot d\vec{s}$$

SUCH VECTOR FIELDS ARE SAID TO BE <u>CONSERVATIVE</u>.

THEOREM

LET $\vec{F}$ BE A C^1 VECTOR FIELD ON $\mathbb{R}^3$ (EXCEPT FOR POSSIBLY FINITELY MANY POINTS.) T.F.A.E.

(1) FOR ANY ORIENTED SIMPLE CLOSED CURVE C:

$$\int_C \vec{F}\cdot d\vec{s} = 0$$

(2) FOR ANY TWO ORIENTED SIMPLE CURVES C_1, C_2 WITH SAME END POINTS

$$\int_{C_1} \vec{F}\cdot d\vec{s} = \int_{C_2} \vec{F}\cdot d\vec{s}$$

(3) THERE EXISTS A FUNCTION f ON $\mathbb{R}^3$ (EXCEPT THE POINTS WHERE $\vec{F}$ IS UNDEFINED) SUCH THAT

$$\vec{F} = \nabla f$$

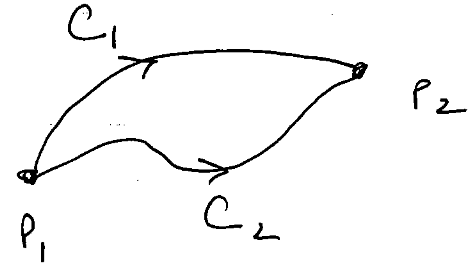
(4) $\nabla \times \vec{F} = \vec{0}$

SUCH FIELDS $\vec{F}$ ARE CALLED <u>CONSERVATIVE FIELDS</u>.

PROOF:

(1) $\Rightarrow$ (2)

LET C_1, C_2 HAVE SAME ENDPOINTS



AND LET $C = C_1 - C_2$, i.e. TRAVERSE C_1 FROM P_1 TO P_2 THEN C_2 FROM P_2 TO P_1. THEN

$$0 = \int_C \vec{F}\cdot d\vec{s} = \int_{C_1} \vec{F}\cdot d\vec{s} - \int_{C_2} \vec{F}\cdot d\vec{s} \qquad (\text{BY } (1))$$

WHENCE

$$\int_{C_1} \vec{F}\cdot d\vec{s} = \int_{C_2} \vec{F}\cdot d\vec{s}$$

SO (2) HOLDS.

(2) $\Rightarrow$ (3)

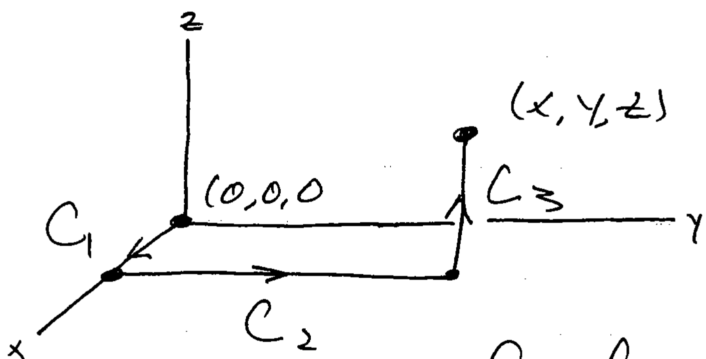
LET C BE ANY ORIENTED PATH FROM $(0,0,0)$ TO (x,y,z). DEFINE

$$f(x,y,z) = \int_C \vec{F}\cdot d\vec{s}$$

OBSERVE THAT $f(x,y,z)$ DOES NOT DEPEND ON THE ACTUAL PATH C BY (2). WE WILL SHOW THAT

$$\nabla f = \vec{F}.$$

TO DO THIS PICK C TO BE THE THREE LINE SEGMENTS BELOW



$$C = C_1 \cup C_2 \cup C_3$$

C_1 : $(t, 0, 0)$ $\quad 0 \le t \le x$
C_2 : $(x, t, 0)$ $\quad 0 \le t \le y$
C_3 : (x, y, t) $\quad 0 \le t \le z$

SUPPOSE $\vec{F} = (F_1, F_2, F_3)$. THEN

$$f(x,y,z) = \int_{C_1} \vec{F}\cdot ds + \int_{C_2} \vec{F}\cdot ds + \int_{C_3} \vec{F}\cdot ds$$

$$= \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$

ONE CAN SEE THAT $\frac{\partial f}{\partial z} = F_3(x,y,z)$ BY THE FUNDAMENTAL THEOREM OF CALCULUS.

BY USING OTHER PATHS ONE CAN SIMILARLY SHOW (EXERCISE) THAT

$$\frac{\partial f}{\partial y} = F_2(x,y,z) \quad \text{AND} \quad \frac{\partial f}{\partial x} = F_1(x,y,z)$$

$\therefore\ \vec{F} = \nabla f$ AS REQUIRED.

$(3) \Rightarrow (4)$

$$\nabla \times (\nabla f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$$= (0,0,0)$$

BY THE EQUALITY OF MIXED 2^{ND} PARTIALS.

$(4) \Rightarrow (1)$

LET S BE ANY SURFACE WITH $\partial S = C$. NOTE S CAN ALWAYS BE CHOSEN SO AS TO AVOID THE EXEPTIONAL POINTS OF $\vec{F}$. THEN

$$\int_C \vec{F} \cdot d\vec{s} \underset{\substack{\uparrow \\ \text{By Stokes Thm}}}{=} \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} \underset{\substack{\uparrow \\ \text{By (4)}}}{=} 0$$

///.

RMKS

- THE FUNCTION f FOR WHICH $\nabla f = \vec{F}$ IS CALLED A (SCALAR) POTENTIAL FOR $\vec{F}$.

- IN CHAPTER 7 WE DISCUSSED A METHOD FOR DETERMINING A POTENTIAL f FOR A CONSERVATIVE FIELD $\vec{F}$, i.e. ANTIDIFFERENTIATE THE EQUATIONS

$$\frac{\partial f}{\partial x} = F_1 \;,\quad \frac{\partial f}{\partial y} = F_2 \;,\quad \frac{\partial f}{\partial z} = F_3$$

AND ADJUST FOR CONSTANTS OF INTEGRATION.

- THE CONSTRUCTION IN PROOF OF (2) $\Rightarrow$ (3) PROVIDES AN ALTERNATIVE METHOD FOR FINDING A POTENTIAL FUNCTION

- THIS THM DOES NOT SPECIALIZE DIRECTLY TO VECTOR FIELDS $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$. IF WE ELIMINATE THE POSSIBILITY OF EXCEPTIONAL POINTS (i.e. $\vec{F}$ MUST BE DEFINED ON ALL OF $\mathbb{R}^2$), THEN THE THEOREM (AND PROOF) GO THROUGH AS BEFORE

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$$

- IF WE ALLOW A VECTOR FIELD $\vec{F}$ ON $\mathbb{R}^2$ TO HAVE FINITELY MANY EXCEPTIONAL POINTS THEN

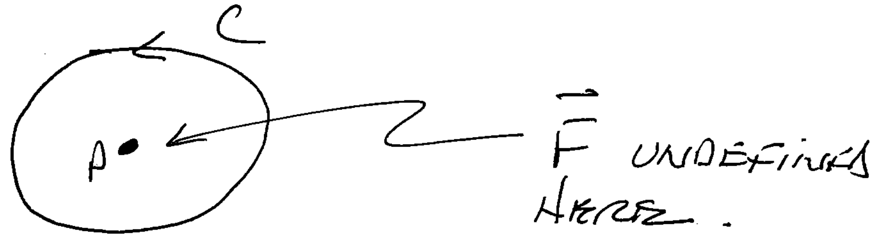
$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$$

BUT (4) DOES NOT IMPLY ANY OF THE OTHERS. i.e. $(4) \not\Rightarrow (1)$.

IF $\vec{F} = (P, Q)$ THEN (4) JUST SAYS

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

THE PROOF BREAKS DOWN IN $(4) \Rightarrow (1)$: IF THE CURVE C ENCIRCLES ONE OF THE EXCEPTIONAL POINTS OF $\vec{F}$,



THEN THE PLANE REGION D WITH $\partial D = C$ NECESSARILY INCLUDES THAT POINT, SO CANNOT APPLY "STOKES" OR MORE PRECISELY VECTOR FORM OF GREENS THM

$$\int_C \vec{F} \cdot d\vec{s} = \iint_D (\nabla \times \vec{F}) \cdot k \, dA = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0$$

CAN'T DO THIS BECAUSE OF POINT P.

- THE RELATIONSHIP BETWEEN div AND curl IS ANALOGOUS TO THAT BETWEEN curl AND grad, i.e. JUST AS

$$\operatorname{curl}(\operatorname{grad}(f)) = \vec{0}$$

Also

$$\operatorname{div}(\operatorname{curl}(\vec{F})) = 0$$

CHECK:

$$\operatorname{div}(\operatorname{curl}(\vec{F})) = \nabla \cdot (\nabla \times \vec{F})$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\,\partial y} - \frac{\partial^2 F_2}{\partial x\,\partial z} + \frac{\partial^2 F_1}{\partial y\,\partial z} - \frac{\partial^2 F_3}{\partial y\,\partial x} + \frac{\partial^2 F_2}{\partial z\,\partial x} - \frac{\partial^2 F_1}{\partial z\,\partial y}$$

$= 0$ BY EQUALITY OF MIXED 2^{nd} PARTIALS.

<u>THEOREM</u>

IF $\vec{F}$ IS A C^1 VECTOR FIELD ON $\mathbb{R}^3$ WITH $\operatorname{div}(\vec{F}) = 0$ THEN THERE EXISTS A C^1 FIELD $\vec{G}$ ON $\mathbb{R}^3$ WITH

$$\vec{F} = \operatorname{curl}(\vec{G})$$

<u>Rmk.</u>

- $\vec{G}$ IS CALLED THE (VECTOR) POTENTIAL OF $\vec{F}$.
- THIS THM. IS ANALOGOUS TO $(4) \Rightarrow (3)$. NOTE NO EXCEPTIONAL POINTS ARE ALLOWED.