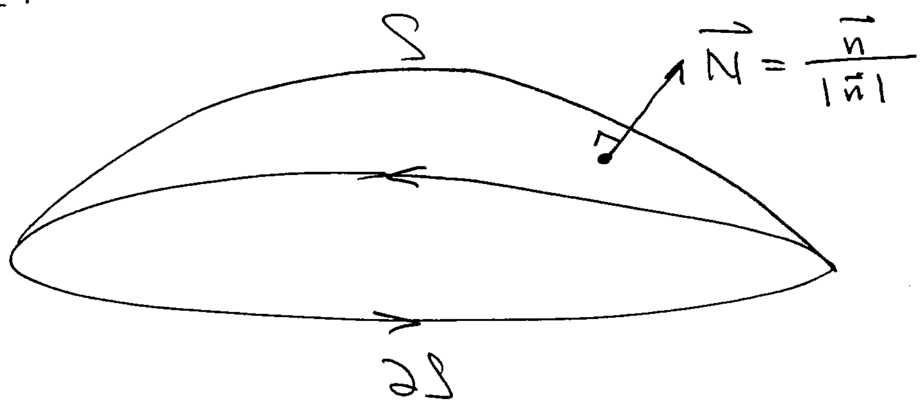
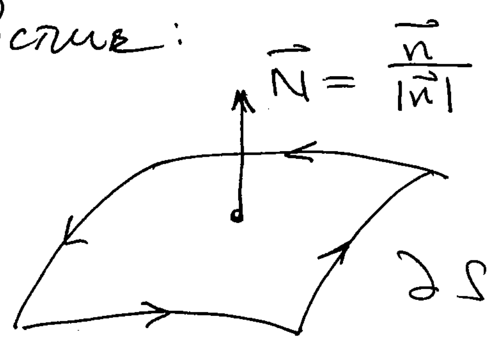
(8.2) STOKES THEOREM

LET S BE AN ORIENTED SURFACE IN $\mathbb{R}^3$. THEN THE ORIENTATION ON S INDUCES AN ORIENTATION ON ITS BOUNDRY CURVE ∂S BY THE FOLLOWING RULE!



AS YOU WALK ALONG ∂S IN ITS ORIENTED DIRECTION YOU MUST SEE THE POSITIVE NORMAL $\vec{n}$ TO S POINTING UP, TO YOUR LEFT

ANOTHER PICTURE:



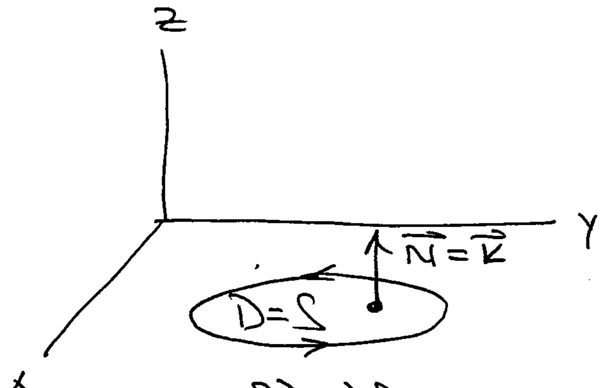
WE CAN NOW STATE STOKES THM FOR SUCH SURFACES.

THM (STOKES'S THM)
LET $\vec{F}(x,y,z)$ BE A C^1 VECTOR FIELD ON $\mathbb{R}^3$ AND S AN ORIENTED SURFACE WITH BOUNDARY ∂S ORIENTED AS ABOVE. THEN

$$\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

RMKS

- NOTE BOTH SIDES DEPEND ON ORIENTATION, SO THE ORIENTATIONS ON S AND ∂S MUST MATCH.
- THIS GENERALIZES ~~[illegible]~~ THE VECTOR FORM OF GREENS THM IN $\mathbb{R}^2$ FOR IF $D\subseteq\mathbb{R}^2$ IS CONSIDERED AS A SURFACE IN $\mathbb{R}^3$, THEN THE UPWARD NORMAL $\vec{N}$ TO THAT SURFACE IS $\vec{k}$



$\partial D=\partial S$

AND $$\iint_S (\nabla\times\vec{F})\cdot dS = \iint_S (\nabla\times F)\cdot\vec{N}\,dS = \iint_D (\nabla\times F)\cdot\vec{k}\,dA$$
$$= \int_{\partial D}\vec{F}\cdot d\vec{s} = \int_{\partial S}\vec{F}\cdot d\vec{s}$$

- ANOTHER AMAZING FACT: IF S_1 AND S_2 ARE TWO SURFACES IN $\mathbb{R}^3$ SUCH THAT $\partial S_1 = \partial S_2$, THEN

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S_1} \vec{F} \cdot d\vec{s}$$

$$= \int_{\partial S_2} \vec{F} \cdot d\vec{s}$$

$$= \iint_{S_2} (\nabla \times \vec{F}) \cdot d\vec{S}$$

i.e. THE FLUX OF $\nabla \times \vec{F}$ ACROSS A SURFACE S DEPENDS ONLY ON ∂S.

<u>Ex.</u> $\vec{F} = (y, -x, e^{xy})$

$\partial S: x^2 + y^2 = 1,\ z = 0$

FIND $\displaystyle\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$

NOTICE THAT S WAS NOT EFFECTIVELY GIVEN, ONLY ∂S WAS

$$\partial S : \begin{cases} x = \cos t \\ y = \sin t \\ z = 0 \end{cases} \qquad 0 \le t \le 2\pi$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$$

$$= \int_{\partial S} y\,dx - x\,dy + e^{xy}\,dz$$

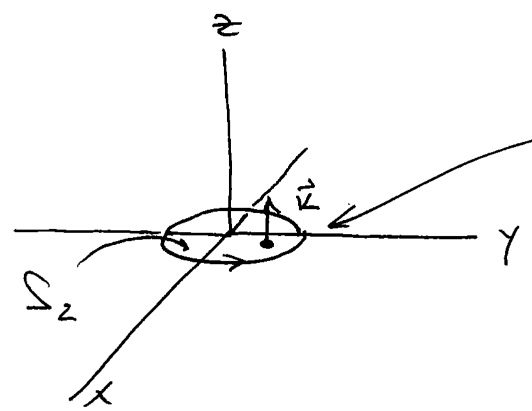
$$= \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t + 0 \right) dt$$

$$= -\int_0^{2\pi} dt = \boxed{-2\pi}$$

ANOTHER WAY: ANY OTHER SURFACE WITH THE SAME BOUNDARY WILL DO, PICK

$$S_2 : \ x^2 + y^2 \le 1, \ z = 0$$

AND NOTE THAT $\partial S_2 = \partial S$.



AREA OF UNIT DISC IS π

OBSERVE THAT ~~A~~ POSITIVE UNIT NORMAL TO S_2 IS $\vec{N} = \vec{k} = (0,0,1)$. ALSO

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & e^{xy} \end{vmatrix} = (xe^{xy}, -ye^{xy}, -2)$$

AND $(\nabla \times \vec{F}) \cdot \vec{N} = -2$. THUS

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_{S_2} (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_{S_2} (\nabla \times F) \cdot \vec{N} \, dS$$

$$= \iint_{S_2} (-2) \, dS = -2 \, A(S_2)$$

$$= \boxed{-2\pi}.$$

RECALL THE PHYSICAL INTERPRETATION OF $\nabla \times \vec{F}$ FROM CHAP. 3.

REGARD $\vec{F}$ AS THE VELOCITY FIELD OF A FLUID MOVING IN $\mathbb{R}^3$. DROP A SMALL PADDLE WHEEL INTO THIS FLUID AT $P \in \mathbb{R}^3$. THEN

- $(\nabla \times F)(P)$ IS PARALLEL TO THE INSTANTANEOUS AXIS OF ROTATION OF THE WHEEL
- $|(\nabla \times F)(P)|$ IS THE INSTANTANEOUS RATE OF ROTATION (IN RADIANS/SEC) ABOUT THIS AXIS

P. 539-541 GIVES A NICE EXPLANATION OF THIS.