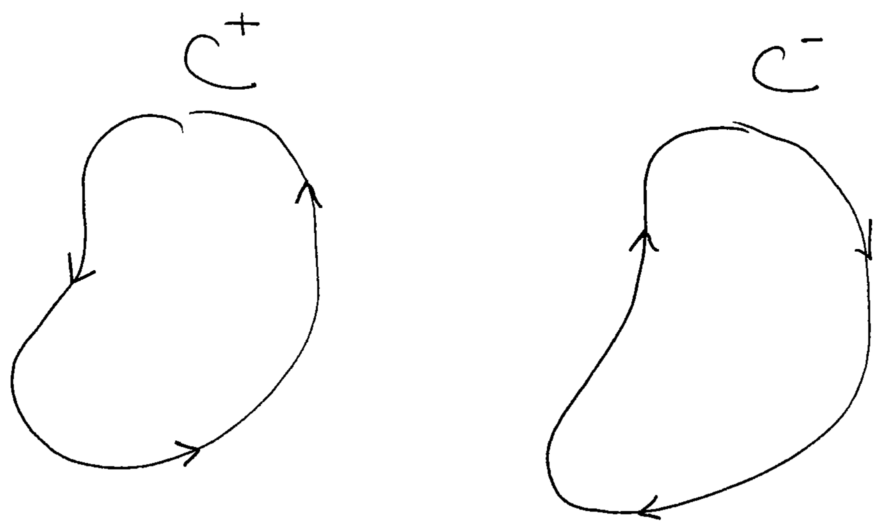
(8.1) GREEN'S THEOREM

GIVEN A CLOSED CURVE C IN THE PLANE WE CONSIDER THE COUNTER-CLOCKWISE DIRECTION TO BE THE POSITIVE DIRECTION ON THE CURVE



GREEN'S THM

LET D BE A REGION IN $\mathbb{R}^2$ AND LET $C = \partial D$. LET $P : D \to \mathbb{R}$ AND $Q : D \to \mathbb{R}$ BE C^1 FUNCTIONS. THEN

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

EX. VERIFY GREEN'S THM. FOR $D: x^2+y^2 \le 1$, $P(x,y) = x$, $Q(x,y) = xy$.

TO COMPUTE LHS WE PARAMETRIZE $C^+ = \partial D$ AS

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \qquad 0 \le t \le 2\pi$$

SO

$$LHS = \int_{C^+} P\,dx + Q\,dy = \int_0^{2\pi} \left(\cos t \cdot (-\sin t) + \cos t \cdot \sin t \cdot (\cos t) \right) dt$$

$$= \int_0^{2\pi} \left(\cos^2 t \cdot \sin t - \cos t \sin t \right) dt$$

$$= -\frac{1}{3}\cos^3 t + \frac{1}{2}\cos^2 t \Big|_0^{2\pi} = 0$$

Also

$$RHS = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_D y\,dx\,dy = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\,dy\,dx$$

$$= \int_{-1}^{1} \frac{1}{2} y^2 \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \frac{1}{2}\int_{-1}^{1} (1-x^2) - (1-x^2)\,dx = \frac{1}{2}\int_{-1}^{1} 0\,dx$$

$$= 0 .$$

<u>Corollary</u>

$$A(D) = \frac{1}{2} \int_{\partial D} x\,dy - y\,dx .$$

PROOF

WE HAVE $P = -y$ AND $Q = x$ SO $\frac{\partial Q}{\partial x} = 1$ AND $\frac{\partial P}{\partial y} = -1$. BY GREEN'S THM

$$\frac{1}{2}\int_{\partial D} x\,dy - y\,dx = \frac{1}{2}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \frac{1}{2}\iint_D (1-(-1))\,dA = \iint_D dA = A(D).$$

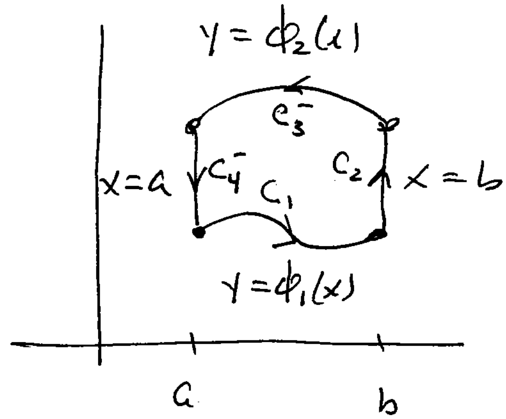
///.

PROOF OF GREEN'S THM.

LEMMA 1: LET D BE y-SIMPLE AND $C = \partial D$ THEN

$$\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dx\,dy \qquad (\text{i.e. } Q = 0).$$

PROOF:



$$C = C_1^+ \cup C_2^+ \cup C_3^- \cup C_4^-$$

THEN

$$\iint_D \frac{\partial P}{\partial y}(x,y)\,dx\,dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx$$

$$= \int_a^b \Big(P(x,\phi_2(x)) - P(x,\phi_1(x)) \Big)\,dx$$

$$= \int_a^b P(t,\phi_2(t))\,dt - \int_a^b P(t,\phi_1(t))\,dt$$

$$= \int_{C_2^+} P\,dx - \int_{C_1^-} P\,dx$$

SINCE C_3 & C_1 ARE PARAMETIZED BY

$C_1^+ : \; (t, \phi_1(t)) \quad a \le t \le b$

$C_3^+ : \; (t, \phi_2(t)) \quad a \le t \le b$

Also

$C_2^+ : \; (b, t) \quad \phi_1(b) \le t \le \phi_2(b) \quad \therefore dx = 0$

$C_4^+ : \; (a, t) \quad \phi_1(a) \le t \le \phi_2(a) \quad \therefore dx = 0$

SO

$$\int_C P\,dx = \int_{C_1^+} P\,dx + \cancel{\int_{C_2^+} P\,dx}^{\,0} + \int_{C_3^-} P\,dx + \cancel{\int_{C_4^-} P\,dx}^{\,0}$$

$$= \int_{C_1^+} P\,dx - \int_{C_3^-} P\,dx$$

$$= -\left(\int_{C_3^+} P\,dx - \int_{C_1^-} P\,dx \right)$$

$$= -\iint_D \frac{\partial P}{\partial y}\,dx\,dy \qquad ///.$$

<u>LEMMA 2</u>

IF D IS x-SIMPLE THEN

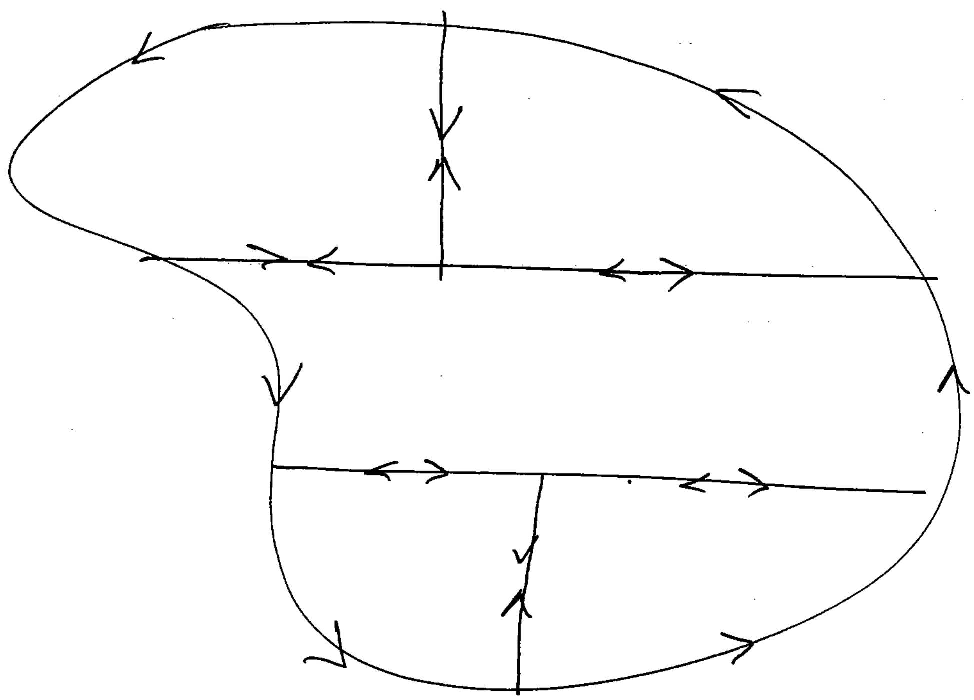
$$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dx\,dy \qquad (\text{i.e. } P=0)$$

EXERCISE: PROVE THIS OR READ P. 521-522

Pf. OF GREEN'S THM: (D BOTH x & y SIMPLE).

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\,dx .$$

IN GENERAL CAN DIVIDE D INTO SIMPLE REGIONS.



LET $\vec{F} = (P, Q)$. THEN GREENS THEOREM CAN BE WRITTEN

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

EXTENDING $\vec{F}$ TO $\mathbb{R}^3$ BY : $\vec{F} = (P, Q, 0)$ THEN

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

SO THAT $(\nabla \times \vec{F}) \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. THUS

<u>THM</u> (VECTOR FORM OF GREENS THM.)

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k} \, dA$$

THIS IS A FORM OF GREEN'S THM WHICH CAN BE READILY GENERALIZED TO $\mathbb{R}^3$.

THERE IS YET ANOTHER FORM OF GREEN'S THM WHICH SUGGESTS ANOTHER GENERALIZATION TO $\mathbb{R}^3$.

LET $c : [a,b] \longrightarrow \mathbb{R}^2$ BE A PARAM. OF A CURVE C IN $\mathbb{R}^2$.

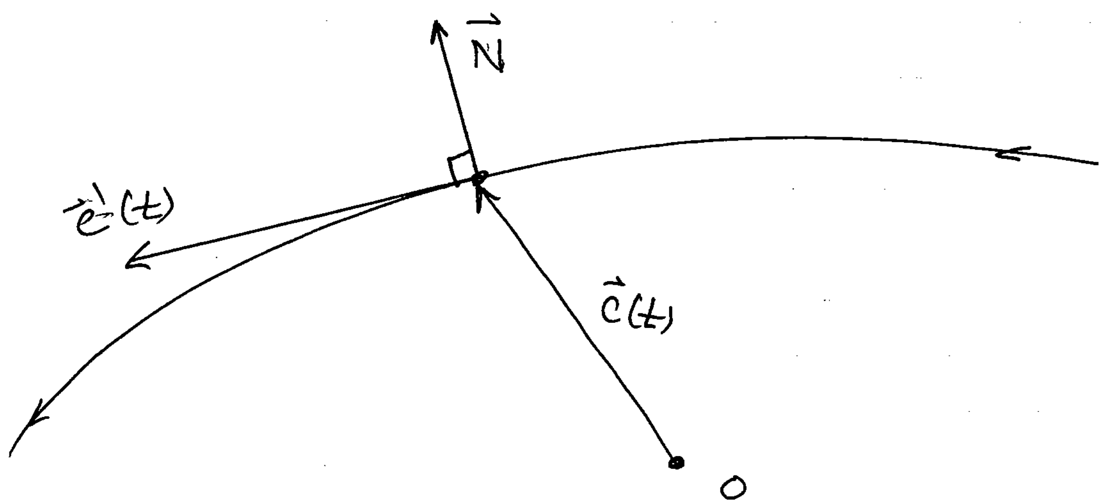
$$\vec{c}(t) = (x(t), y(t)).$$

THEN THE TANGENT VECTOR AT $\vec{c}(t)$ IS

$$\vec{c}\,'(t) = (x'(t), y'(t))$$

AND A NORMAL VECTOR TO C AT $c(t)$ IS

$$(y'(t), -x'(t))$$

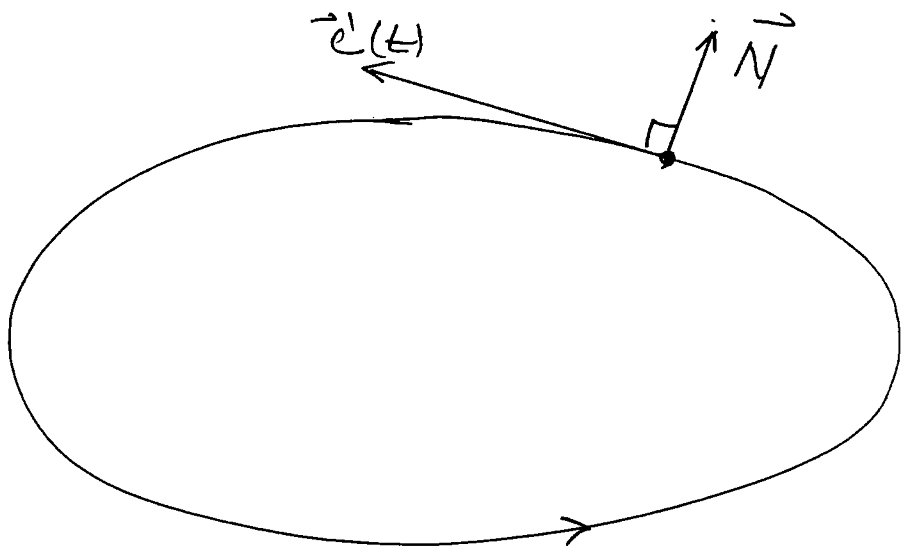


AND

$$\vec{N} = \frac{(y(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}$$

IS A <u>UNIT</u> NORMAL TO C AT $\vec{c}(t)$.

NOTE: IF C IS A CLOSED CURVE, THEN $\vec{N}$ ABOVE IS AN OUTWARD UNIT NORMAL TO C AT $\vec{c}(t)$



CHECK: $(x', y') \cdot (y', -x') = x'y' - x'y' = 0$
SO $c' \perp \vec{N}$. FROM THE ABOVE PICTURE WE SEE

$$c' = (-, +) \quad \& \quad \vec{N} = (+, +)$$

SO THE SIGNS APPEAR TO BE RIGHT.

THM (DIVERGENCE THM IN $\mathbb{R}^2$)

$$\int_{\partial D} \vec{F} \cdot \vec{N} \, ds = \iint_D \operatorname{div} \vec{F} \, dA$$

PROOF: LET $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
THEN

$$\int_{\partial D} \vec{F} \cdot \vec{N}\, ds = \int_a^b \frac{P(c(t))\,y'(t) - Q(c(t))\,x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} \cdot \|c'(t)\|\, dt$$

$$= \int_a^b -Q(\vec{c}(t))\frac{dx}{dt} + P(c(t))\frac{dy}{dt}$$

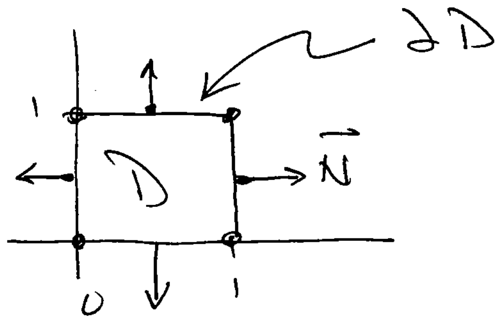
$$= \int_{\partial D} -Q\,dx + P\,dy$$

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA \qquad \text{BY GREEN'S THM}$$

$$= \iint_D \operatorname{div}(\vec{F})\, dA.$$

///.

EX COMPUTE THE INTEGRAL OF THE NORMAL COMPONENT OF $\vec{F} = (x^2, xy)$ ~~ABOUT THE~~ AROUND THE UNIT SQUARE $[0,1] \times [0,1] = D$



$$\operatorname{div}(\vec{F}) = 2x + x = 3x$$

PHYSICALLY THIS IS THE FLUX OF A FLUID IN $\mathbb{R}^2$ WITH VELOCITY FIELD $\vec{F}$ FLOWING OUT OF D

HARD WAY

$$\int_{\partial D} \vec{F} \cdot N \, ds = \text{FOUR TERMS} \ldots.$$

EASY WAY

$$\int_{\partial D} \vec{F} \cdot \vec{N} \, ds = \iint_D \operatorname{div}(\vec{F}) \, dA = \int_0^1 \int_0^1 3x \, dx \, dy$$

$$= \frac{3x^2}{2} \Big|_0^1 = \boxed{\frac{3}{2}}.$$