(7.6) SURFACE INTEGRALS OF VECTOR FIELDS.

LET $\vec{F}(x,y,z)$ BE A VECTOR FIELD ON $\mathbb{R}^3$ AND S A SURFACE PARAMETRIZED BY $\Phi : D \to \mathbb{R}^3$. THE <u>INTEGRAL OF F OVER S</u> IS

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D F \cdot (T_u \times T_v)\, du\, dv$$

<u>Rmk</u>: WE will show THAT THE INTEGRAL ON THE RHS IS INDEPENDENT OF THE PARAMETRIZATION Φ, i.e. IF $\Psi : D^* \to \mathbb{R}^3$ IS ANOTHER PARAMETRIZATION OF S AND

$$T_u^* = \frac{\partial \Psi}{\partial u}$$

$$T_v^* = \frac{\partial \Psi}{\partial v}$$

THEN
$$\iint_{D^*} \vec{F}(\Psi(u,v)) \cdot (T_u^* \times T_v^*)\, du\, dv$$

$$= \iint_D \vec{F}(\Phi(u,v)) \cdot (T_u \times T_v)\, du\, dv$$

RMK: RECALL IF $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$ THEN WE CALCULATED

$$T_u \times T_v = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right)$$

EX LET S BE THE UNIT SPHERE ABOUT $(0,0,0)$ PARAMETRIZED BY

$$\Phi(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

FOR $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. LET

$$\vec{F}(x,y,z) = (x, y, z).$$

THEN

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\Phi(\phi,\theta)) \cdot (T_\phi \times T_\theta)\, d\phi\, d\theta$$

(ANS: $+4\pi$)

NOW

$$\frac{\partial(y,z)}{\partial(\phi,\theta)} = \begin{vmatrix} \cos\phi\sin\theta & \sin\phi\cos\theta \\ -\sin\phi & 0 \end{vmatrix} = \sin^2\phi\cos\theta$$

$$\frac{\partial(z,x)}{\partial(\phi,\theta)} = \begin{vmatrix} -\sin\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi\sin\theta \end{vmatrix} = \sin^2\phi\sin\theta$$

$$\frac{\partial(x,y)}{\partial(\phi,\theta)} = \begin{vmatrix} \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \cos\phi\sin\theta & \sin\phi\cos\theta \end{vmatrix} = \cos\phi\sin\phi$$

SO

$$T_\phi \times T_\theta = \sin\phi \left(\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi\right)$$

AND

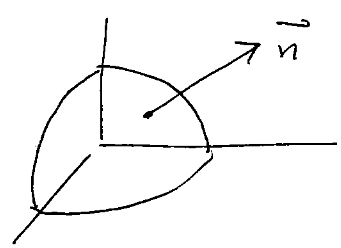
$$\vec{F}(\Phi(\phi,\theta)) \cdot (T_\phi \times T_\theta) = \sin\phi$$

THUS

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta = 2\pi(-\cos\phi)\Big|_0^{\pi}$$

$$= -2\pi(-1-1) = \boxed{4\pi}$$

<u>NOTE</u>:
$T_\phi \times T_\theta$ ABOVE IS THE OUTWARD POINTING NORMAL VECTOR TO THE SPHERE S.

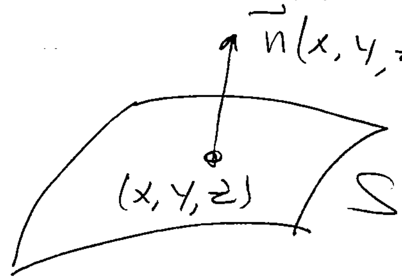


IF WE HAD SWAPPED THE ORDER OF THE VARIABLE ϕ AND θ AND COMPUTED $T_\theta \times T_\phi$ INSTEAD, OUR ANSWER WOULD HAVE DIFFERED BY A SIGN. i.e. -4π (SEE EX 1 ON P. 484)

THUS $\iint_S \vec{F} \cdot d\vec{S}$ DEPENDS ON THE SURFACE S, TOGETHER WITH AN ORIENTATION OF S

DEFN:
AN ORIENTED SURFACE IS A 2-SIDED SURFACE IN WHICH ONE SIDE HAS BEEN SPECIFIED AS THE OUTWARD OR POSITIVE SIDE.

TO SPECIFY AN ORIENTATION OF S, WE NEED ONLY SPECIFY AN OUTWARD POINTING NORMAL VECTOR $\vec{n}$ WHICH VARIES CONTINUOUSLY



NOTE!
NOT ALL SURFACES ARE TWO-SIDED!
SEE P. 486, MÖBIUS STRIP.

SUCH SURFACES ARE CALLED NON-ORIENTABLE.

BY CONVENTION, WE WILL TAKE THE OUTWARD DIRECTION ON A GRAPH $S: z = g(x,y)$ TO HAVE NORMAL VECTOR $\vec{n}$ WITH POSITIVE $\vec{k}$ COMPONENT, i.e. UPWARD POINTING.

RECALL IF S IS GIVEN BY $z = g(x,y)$ THEN

$$\vec{n} = T_x \times T_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$$

IS THE UPWARD POINTING NORMAL. THUS IN THIS CASE

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(x,y,g(x,y)) \cdot \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) dx\,dy$$

$$= \iint_D \left(-\frac{\partial g}{\partial x} F_1 - \frac{\partial g}{\partial y} F_2 + F_3\right) dx\,dy$$

EX. FIND INTEGRAL OF $\vec{F} = (x+z, 0, 0)$ OVER S : $z = x^2+y^2$ LYING OVER THE SQUARE $D = [0,1] \times [0,1]$ IN xy-PLANE

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-2x)(x+x^2+y^2)\, dx\,dy$$

$$= \ldots\ldots = \boxed{-\frac{3}{2}}$$

WE SAY $\Phi : D \to \mathbb{R}^3$ IS AN ORIENTATION PRESERVING PARAMETRIZATION OF $S = \text{image}(\Phi)$ IF

$$n = T_u \times T_v \qquad \Big(\text{WHERE } \underset{\substack{\uparrow \\ 1^{ST}}}{\Phi}(u, \underset{\substack{\uparrow \\ 2^{ND}}}{v})\Big)$$

IS AN <u>OUTWARD</u> NORMAL TO S.

<u>Thm</u>:
LET $\Phi_1(u_1,v_1)$ AND $\Phi_2(u_2,v_2)$ BE TWO REGULAR ORIENTATION PRESERVING PARAMETRIZATIONS OF S. THEN

$$\iint_{D_1} \vec{F}(\Phi_1(u_1,v_1)) \cdot (T_{u_1} \times T_{v_1})\, du_1\, dv_1$$

$$= \iint_{D_2} \vec{F}(\Phi_2(u_2,v_2)) \cdot (T_{u_2} \times T_{v_2})\, du_2\, dv_2$$

PROOF DEPENDS ON CHANGE OF VARIABLE FORMULA. (OMITTED.)

WE CAN RELATE THE INTEGRAL OF $\vec{F}$ OVER S TO A SCALAR FUNCTION AS FOLLOWS:

IF Φ IS AN O.P. PARAM. THEN

$$\vec{N} = \frac{\vec{n}}{|\vec{n}|} = \frac{T_u \times T_v}{|T_u \times T_v|}$$

IS AN OUTWARD POINTING UNIT NORMAL TO S AND

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (T_u \times T_v)\, du\,dv$$

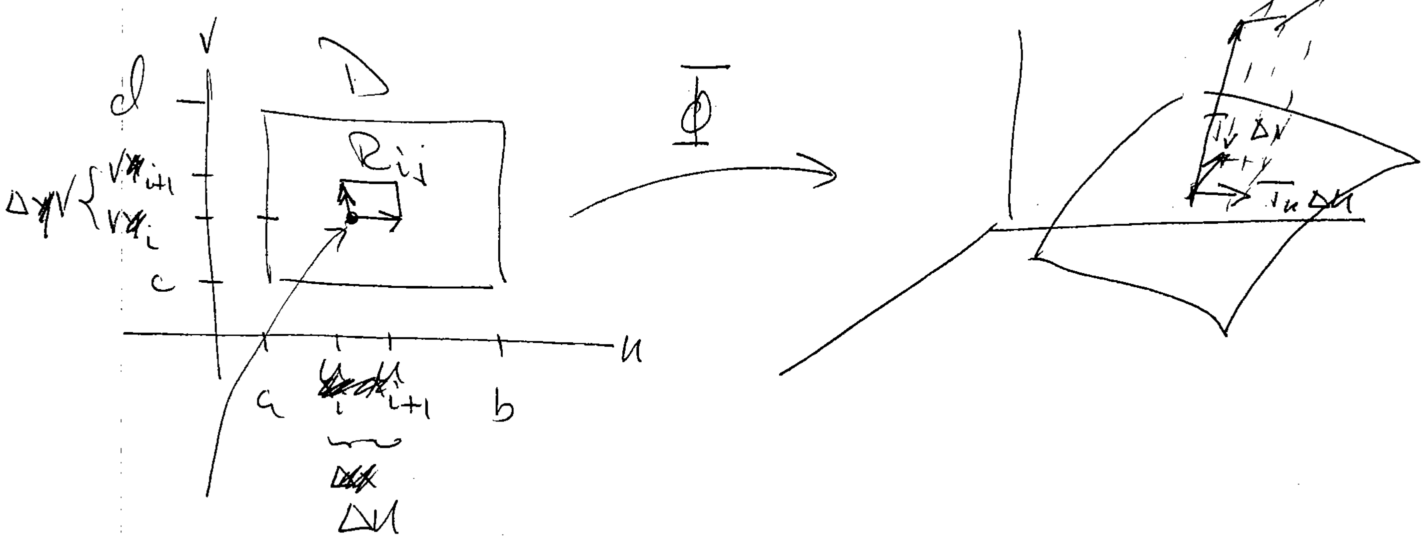
$$= \iint_D \left(F \cdot \frac{(T_u \times T_v)}{|T_u \times T_v|} \right) |T_u \times T_v|\, du\,dv$$

$$= \iint_D (F \cdot \vec{N})\, |T_u \times T_v|\, du\,dv$$

$$= \iint_S (\vec{F} \cdot \vec{N})\, dS$$

PHYSICAL INTERPRETATION

LET $\vec{F}$ BE THE VELOCITY FIELD OF A FLUID IN $\mathbb{R}^3$.



THE IMAGE $\Phi(R_{ij})$ IS APPROX. BY THE ~~PPIPED~~ PARALLELOGRAM SPANNED BY

$T_u \Delta u$ & $T_v \Delta v$

ALSO

$$\vec{F}(\Phi(u_i, v_j)) \cdot (\Delta u\, T_u \times \Delta v\, T_v)$$

$$= \vec{F} \cdot (T_u \times T_v)\, \Delta u\, \Delta v$$

≈ THE VOL. OF THE PPIPED SPANNED BY $\vec{F}$, $\Delta u\, T_u$, $\Delta v\, T_v$.

THIS IS ALSO THE VOL. OF FLUID FLOWING THROUGH PGRAM PER UNIT TIME.

LET $T(x,y,z)$ BE THE TEMPERATURE AT (x,y,z). CLASSICAL PHYSICISTS THOUGHT OF HEAT ENERGY AS BEING A TYPE OF 'FLUID' WHICH FLOWS FROM ONE POINT TO ANOTHER. IT NATURALLY FLOWS IN THE DIRECTION OF COLDER POINTS.

THUS

$$\vec{F}(x,y,z) = -K \nabla T(x,y,z)$$

IS THE VELOCITY FIELD OF THIS FLUID.

(HERE K IS A POSITIVE CONSTANT DEPENDING ON THE UNITS INVOLVED.)

(RECALL ∇T POINTS IN DIR OF FASTEST INCREASE OF T, SO $-\nabla T$ POINTS IN DIR. OF FASTEST DECREASE.

IN THIS CASE

$$\iint_S \vec{F} \cdot d\vec{S}$$

IS THE <u>RATE OF HEAT FLOW</u> OR <u>HEAT FLUX</u> ACROSS THE SURFACE S.

EX Let $T(x,y,z) = 2x^2 + 3y^2 + z^2$, and let ~~...~~ $K = 1$. Find the heat flux across the sphere of radius R about $(0,0,0)$:

$$S: x^2 + y^2 + z^2 = R^2$$

$$\vec{F} = -\nabla T = (-4x, -6y, -2z)$$

$$S: \begin{cases} x = R\sin\phi\cos\theta \\ y = R\sin\phi\sin\theta \\ z = R\cos\phi \end{cases}$$

Exercise: $|T_\phi \times T_\theta| = R^2\sin\phi$

Also $\vec{N} = \frac{1}{R}(x, y, z)$ is unit (outer) normal.

$$\therefore \vec{F}\cdot\vec{N} = \frac{1}{R}(-4x^2 - 6y^2 - 2z^2) = \left(-\frac{2}{R}\right)(2x^2 + 3y^2 + z^2)$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = \left(-\frac{2}{R}\right)\iint_S (2x^2+3y^2+z^2)\,dS$$

$$= \left(-\frac{2}{R}\right)\int_0^{2\pi}\int_0^{\pi} R^2\left(2\sin^2\phi\cos^2\theta + 3\sin^2\phi\sin^2\theta + \cos^2\phi\right)R^2\sin\phi\,d\phi\,d\theta$$

$$= -2R^3\int_0^{2\pi}\int_0^{\pi}(\qquad\qquad)\,d\phi\,d\theta \quad < 0.$$

IF $\vec{E}(x,y,z)$ IS AN ELECTRIC FIELD THEN

$$\iint_S \vec{E}\cdot d\vec{S}$$

IS CALLED THE <u>ELECTRIC FLUX</u> ACROSS S. IF S IS A <u>CLOSED SURFACE</u>

THEN GAUSS LAW SAYS

$$\iint_S \vec{E}\cdot d\vec{S} = Q$$

WHERE Q IS THE NET CHARGE ENCLOSED BY THE SURFACE.

THIS PROVIDES YET ANOTHER WAY TO PHRASE THESE PROBLEMS.

(7.7) READ CURVATURE: P. 500 – 504.

DO 7.7 ~~#~~ 4, 6, 8.

REVIEW PROBS FOR CHAP 7

P. 514:

1acd, 2ac, 3b, 4, 6, 7ac, 8, 9, 11, 12a

13, 14, 18, 19, 23,