(7.5) SURFACE INTEGRALS OF SCALAR FUNCTIONS

LET $f: \mathbb{R}^3 \to \mathbb{R}$ BE CONTINUOUS AND LET $\Phi : D \to \mathbb{R}^3$ BE A PARAMETRIZATION OF A SMOOTH SURFACE S.

DEFN:
THE INTEGRAL OF f OVER S IS

$$\iint_S f \, dS = \iint_D f(\Phi(u,v)) \, |T_u \times T_v| \, du \, dv$$

THUS IF $f = 1$, WE GET AREA OF S.

NOTE: $\iint_S f \, dS$ IS INDEPENDENT OF THE CHOICE OF PARAMETRIZATION. (PROOF LATER)

EX HELICOID

$$S \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \theta \end{cases} \qquad \begin{array}{l} \text{WHERE} \\ 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array}$$

LET $f(x,y,z) = \sqrt{x^2+y^2+1}$. DETERMINE $\iint_S f \, dS$.

AS BEFORE $\frac{\partial(y,z)}{\partial(r,\theta)} = \sin\theta$, $\frac{\partial(z,x)}{\partial(r,\theta)} = -\cos\theta$, $\frac{\partial(x,y)}{\partial(r,\theta)} = r$

AND $|T_r \times T_\theta| = \sqrt{1+r^2}$. ALSO

$$f(r\cos\theta, r\sin\theta, \theta) = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 1} = \sqrt{1+r^2}$$

SO

$$\iint_S f\,dS = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\cdot\sqrt{1+r^2}\,dr\,d\theta$$

$$= 2\pi \int_0^1 (1+r^2)\,dr = 2\pi\left(r+\tfrac{1}{3}r^3\right)\Big|_0^1$$

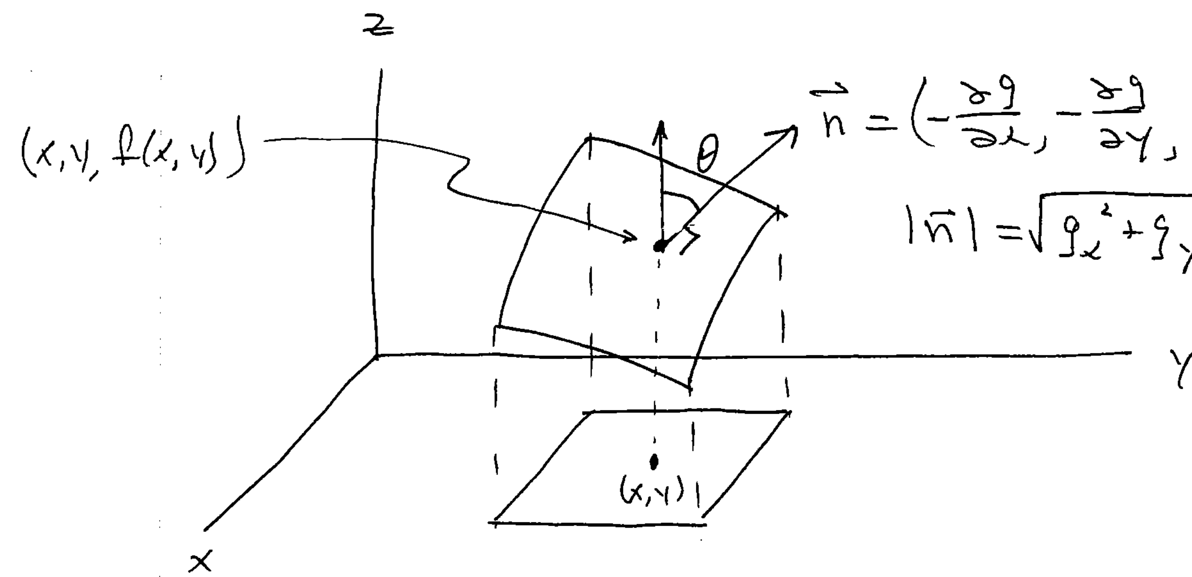
$$= 2\pi\left(1+\tfrac{1}{3}\right) = 2\pi\cdot\frac{4}{3} = \boxed{\frac{8\pi}{3}}$$

IF S IS THE GRAPH $z = g(x,y)$ WE'VE SEEN THAT

$$|T_x \times T_y| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

WHENCE

$$\iint_S f\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\,dx\,dy$$



$$\vec{n} = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$$

$$|\vec{n}| = \sqrt{g_x^2 + g_y^2 + 1}$$

LET $\vec{k} = (0,0,1)$ BE THE UNIT VECTOR ALONG THE z AXIS AND

$$\vec{N} = \frac{\vec{n}}{|\vec{n}|} = \frac{(-g_x, -g_y, 1)}{\sqrt{g_x^2 + g_y^2 + 1}}$$

THE UNIT NORMAL TO S AT $(x, y, g(x,y))$. THEN

$$\vec{N} \cdot \vec{k} = \cos\theta$$

WHERE $\theta = \theta(x,y)$ IS THE ANGLE BETWEEN $\vec{N}$ AND $\vec{k}$ (OR $\vec{n}$ AND $\vec{k}$). ALSO

$$\vec{N} \cdot \vec{k} = \frac{1}{\sqrt{g_x^2 + g_y^2 + 1}}.$$

THUS $|T_x \times T_y| = |\vec{n}| = \sqrt{g_x^2 + g_y^2 + 1} = \dfrac{1}{\cos\theta}$.

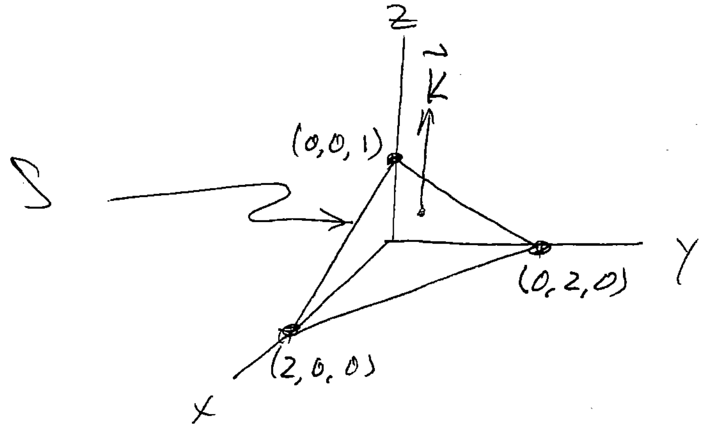
WHENCE

$$= \iint_D \frac{f(x,y,g(x,y))}{\cos\theta}\,dx\,dy\,.$$

THIS FORMULA IS PARTICULARLY HELPFUL WHEN $\theta = \theta(x,y) = \text{const.}$

Ex COMPUTE $\iint_S y\,dS$ WHERE S IS

THE PORTION OF THE PLANE $x+y+2z=2$ LYING IN THE 1^{st} OCTANT. $(x\geq 0, y\geq 0, z\geq 0)$

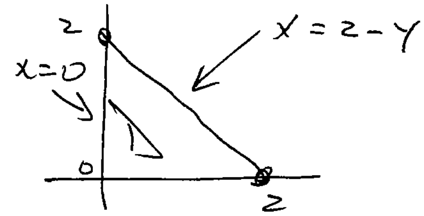


SO $\vec{n}=(1,1,2)$ IS NORMAL TO S AND $|\vec{n}| = \sqrt{1+1+4} = \sqrt{6}$, SO $\vec{N} = \frac{1}{\sqrt{6}}(1,1,2)$ IS A UNIT NORMAL. THUS

$$\cos\theta = \vec{N}\cdot\vec{k} = \frac{1}{\sqrt{6}}(1,1,2)\cdot(0,0,1) = \frac{2}{\sqrt{6}}$$

AND

$$\iint_S y\,dS = \iint_D \frac{y}{\left(\frac{2}{\sqrt{6}}\right)}\,dx\,dy$$



$$= \frac{\sqrt{6}}{2}\int_0^2\int_0^{2-y} y\,dx\,dy = \frac{\sqrt{6}}{2}\int_0^2 y(2-y)\,dy = \frac{\sqrt{6}}{2}\int_0^2 (2y-y^2)\,dy$$

$$= \frac{\sqrt{6}}{2}\left(y^2-\frac{1}{3}y^3\right)\Big|_0^2 = \frac{\sqrt{6}}{2}\left(4-\frac{8}{3}\right) = \frac{\sqrt{6}}{2}\cdot\frac{4}{3} = \boxed{\frac{2\sqrt{6}}{3}}$$