(7.4) AREA OF A PARAMETRIZED SURFACE

SUPPOSE $D \subseteq \mathbb{R}^2$, $\Phi : D \to \mathbb{R}^3$ AND THAT

(i) D IS AN ELEMENTARY REGION

(ii) Φ IS ONE-TO-ONE AND C^1 (EXCEPT POSSIBLY AT BOUNDARY POINTS)

(iii) $S = \text{image}(\Phi) = \Phi(D)$ IS REGULAR (EXCEPT POSSIBLY AT FINITELY MANY POINTS.)

DEFN:
THE SURFACE AREA OF S IS

$$A(S) = \iint_D |T_u \times T_v| \, du\, dv$$

IF S IS THE UNION OF SUCH REGULAR SURFACES, THEN ITS AREA IS THE SUM OF THE AREAS OF THE REGULAR COMPONENTS.

RECALL

$$T_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$$

$$T_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

AND

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \left(\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, -\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \right)$$

$$= \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right)$$

SO

$$|T_u \times T_v| = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}$$

SO

$$A(S) = \iint_D \sqrt{(\quad)^2 + (\quad)^2 + (\quad)^2}\, du\, dv$$

EX. THE TORUS ABOUT $x^2+y^2=R^2,\ z=0\ (R>1)$.

$$\begin{cases} x = (R+\cos\phi)\cos\theta \\ y = (R+\cos\phi)\sin\theta \\ z = \sin\phi \end{cases}$$

WHERE
$0 \le \phi \le 2\pi$
$0 \le \theta \le 2\pi$

$$\frac{\partial(y,z)}{\partial(\phi,\theta)} = \begin{vmatrix} \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{vmatrix} = \begin{vmatrix} -\sin\phi\sin\theta & (R+\cos\phi)\cos\theta \\ \cos\phi & 0 \end{vmatrix} = -(R+\cos\phi)\cos\phi\cos\theta$$

$$\frac{\partial(z,x)}{\partial(\phi,\theta)} = \begin{vmatrix} \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \\ \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\phi & 0 \\ -\sin\phi\cos\theta & -(R+\cos\phi)\sin\theta \end{vmatrix} = -(R+\cos\phi)\cos\phi\sin\theta$$

$$\frac{\partial(x,y)}{\partial(\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} -\sin\phi\cos\theta & -(R+\cos\phi)\sin\theta \\ -\sin\phi\sin\theta & (R+\cos\phi)\cos\theta \end{vmatrix}$$

$$= -(R+\cos\phi)\sin\phi\cos^2\theta - (R+\cos\phi)\sin\phi\sin^2\theta$$

$$= -(R+\cos\phi)\sin\phi$$

$$\sqrt{[\ ]^2+[\ ]^2+[\ ]^2} = \sqrt{(R+\cos\phi)^2\cos^2\phi\cos^2\theta+(R+\cos\phi)^2\cos^2\phi\sin^2\theta+(R+\cos\phi)^2\sin^2\phi}$$

$$= \sqrt{(R+\cos\phi)^2(\cos^2\phi+\sin^2\phi)} = R+\cos\phi$$

$$A = \int_0^{2\pi}\int_0^{2\pi} R+\cos\phi \, d\phi\, d\theta = 2\pi\cdot(R\phi+\sin\phi)\Big|_0^{2\pi} = 2\pi\cdot 2\pi\cdot R$$

$\cancel{A = 2\pi}$ $$\boxed{A = 4\pi^2 R}$$

Ex Helicoid

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \theta \end{cases} \qquad \begin{array}{l} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array}$$

$$\frac{\partial(y,z)}{\partial(r,\theta)} = \begin{vmatrix} \sin\theta & r\cos\theta \\ 0 & 1 \end{vmatrix} = \sin\theta$$

$$\frac{\partial(z,x)}{\partial(r,\theta)} = \begin{vmatrix} 0 & 1 \\ \cos\theta & -r\sin\theta \end{vmatrix} = -\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\sqrt{(\ )^2 + (\ )^2 + (\ )^2} = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2}$$

$$A = \int_0^{2\pi}\int_0^1 (1+r^2)^{1/2}\,dr\,d\theta = 2\pi\left[\frac{r}{2}(1+r^2)^{1/2} + \frac{1}{2}\ln\left|r + (1+r^2)^{1/2}\right|\right]_0^1$$

$$= 2\pi\left[\frac{1}{2}\cdot\sqrt{2} + \frac{1}{2}\ln|1+\sqrt{2}|\right]$$

$$= \pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right)$$

RECALL IF S IS A GRAPH $z = f(x,y)$, THEN

$$T_x \times T_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

SO

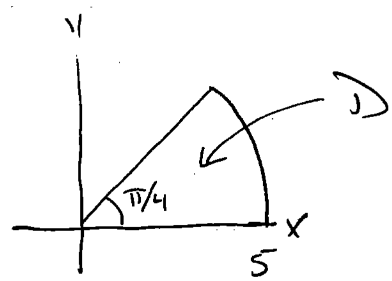
$$|T_x \times T_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

AND

$$A(S) = \iint_D \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dx\,dy$$

WHERE D IS THE DOMAIN OF f IN THE xy PLANE.

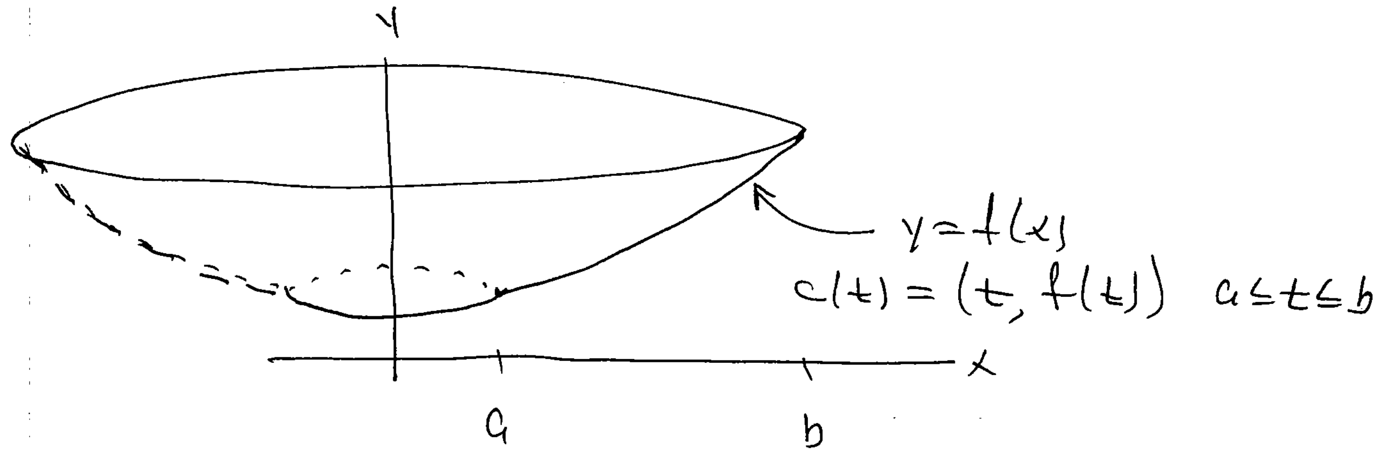
<u>EX</u> FIND SURFACE AREA UNDER $z = x^2 + y^2$ OVER $D = \{(x,y) \mid 0 \le y \le x,\ x^2+y^2 \le 25\}$



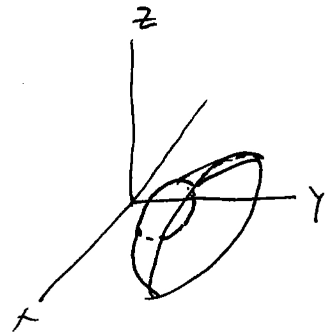
ANS:

$$A(S) = \frac{\pi}{16}\left(101\sqrt{101} - 1\right)$$

LET S BE A SURFACE OF REVOLUTION OBTAINED BY ROTATING THE GRAPH $y = f(x)$ ABOUT THE y-AXIS, FOR $a \le x \le b$ (ASSUME $a \ge 0$)



S IS PARAMETRIZED BY



$$\begin{cases} x = t\cos\theta \\ y = f(t) \\ z = t\sin\theta \end{cases} \qquad \begin{array}{l} \text{FOR} \\ a \le t \le b \\ 0 \le \theta \le 2\pi \end{array}$$

$$\frac{\partial(y,z)}{\partial(t,\theta)} = \begin{vmatrix} f'(t) & 0 \\ \sin\theta & t\cos\theta \end{vmatrix} = t f'(t)\cos\theta$$

$$\frac{\partial(z,x)}{\partial(t,\theta)} = \begin{vmatrix} \sin\theta & t\cos\theta \\ \cos\theta & -t\sin\theta \end{vmatrix} = -t$$

$$\frac{\partial(x,y)}{\partial(t,\theta)} = \begin{vmatrix} \cos\theta & -t\sin\theta \\ f'(t) & 0 \end{vmatrix} = t f'(t)\sin\theta$$

$$A(S) = \int_0^{2\pi}\int_a^b \left(f'(t)^2 t^2\cos^2\theta + f'(t)^2 t^2\sin^2\theta + t^2\right)^{1/2} dt\,d\theta$$

$$= \int_0^{2\pi} \int_a^b t\left(f'(t)^2 + 1\right)^{1/2} dt\, d\theta$$

$$\boxed{A(S) = 2\pi \int_a^b x \sqrt{1 + f'(x)^2}\, dx}$$

IF WE REMOVE THE RESTRICTION THAT $a \geq 0$ WE GET

$$A(S) = 2\pi \int_a^b |x| \sqrt{1 + f'(x)^2}\, dx$$

(EXERCISE.)

IF WE REVOLVE $y = f(x)$ ABOUT THE x-AXIS WE GET

$$A(S) = 2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2}\, dx$$

(EXERCISE, OR READ P. 466.)

READ JUSTIFICATION OF FORMULA FOR $A(S)$ IN TERMS OF RIEMANN SUMS ON P. 462-463.