(4.3) PARAMETRIZED SURFACES

LET $D \subseteq \mathbb{R}^2$ AND $\Phi : D \to \mathbb{R}^3$ BE A C^1 MAPPING. i.e.

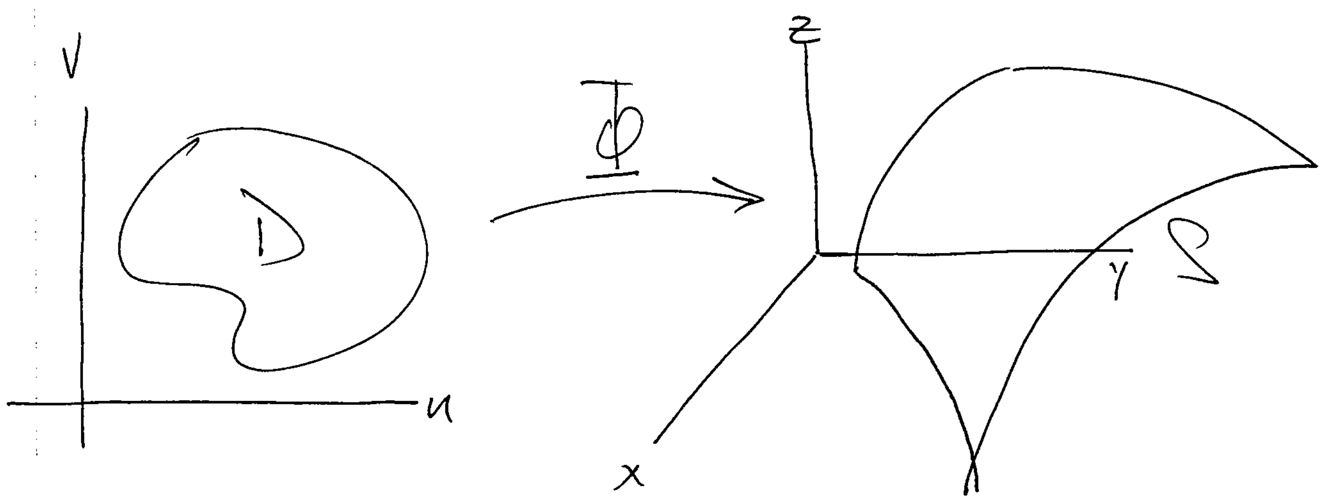
$$\Phi(u,v) = \big(x(u,v),\, y(u,v),\, z(u,v)\big)$$

WHERE THE PARTIAL DERIVATIVES OF x, y, AND z WITH RESPECT TO u AND v, EXIST AND ARE CONTINUOUS.

WE CALL

$$S = \text{image}(\Phi) = \Phi(D)$$

A ~~[illegible]~~ <u>C^1 SURFACE</u> AND Φ A <u>PARAMETRIZATION OF S</u>.



NOTE: S MAY NOT BE THE GRAPH OF ANY FUNCTION $f : \mathbb{R}^2 \to \mathbb{R}$,

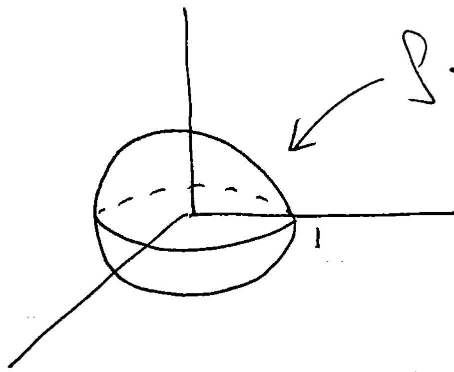
$$S : z = f(x,y)$$

$\underline{\text{Ex.}}$ $\Phi : [0,\pi] \times [0,2\pi] \longrightarrow \mathbb{R}^3$

$$\Phi(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

OBSERVE THAT THIS IS JUST SPHERICAL COORDS WITH $\rho = 1$ FIXED.

$S = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$

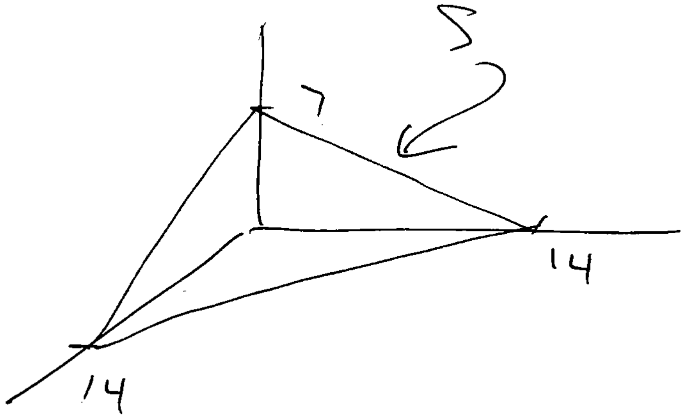


$\underline{\text{Ex.}}$ $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

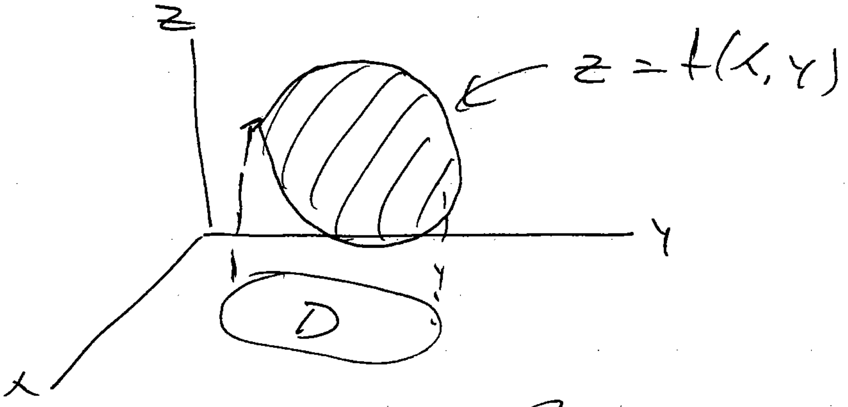
$$\Phi(u,v) = (u, v, -\tfrac{1}{2}u - \tfrac{1}{2}v + 7) \qquad (u,v) \in \mathbb{R}^2$$

$S = \Phi(\mathbb{R}^2)$ IS THE PLANE $x + y + 2z = 14$

S

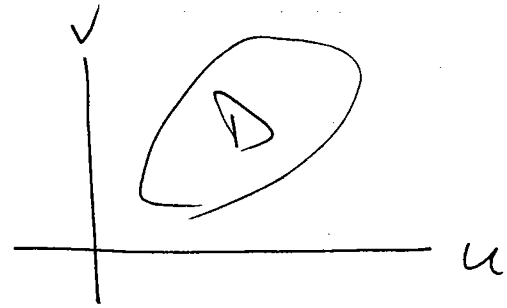


EX. IF S IS THE GRAPH $z=f(x,y)$ FOR $(x,y)\in D\subseteq\mathbb{R}^2$



THEN S CAN BE PARAMETRIZED BY

$$\Phi(u,v)=(u,v,f(u,v)) \quad \text{for } (u,v)\in D$$



EX. $D=[0,2\pi]\times[0,2\pi]$ IN ϕ,θ PLANE

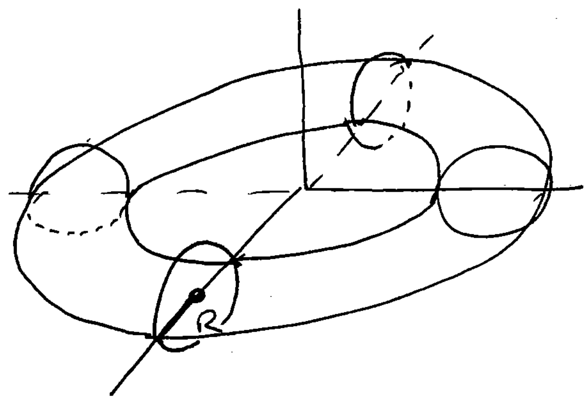
$\Phi: D\to\mathbb{R}^3$ GIVEN BY

FIX $R>1$

$$\Phi(\phi,\theta)=((R+\cos\phi)\cos\theta,\ (R+\cos\phi)\sin\theta,\ \sin\phi)$$

NOTE: ϕ,θ ARE NOT ~~[illegible]~~ RELATED TO SPHERICAL COORDS.

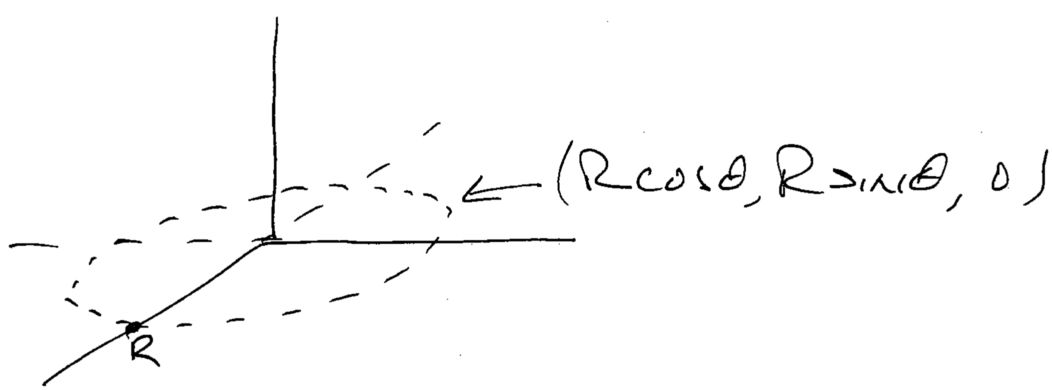
FOR $\theta = 0$ GET $(R + \cos\phi, 0, \sin\phi)$ $\quad 0 \le \phi \le 2\pi$



FOR $\phi = 0$ GET $((R+1)\cos\phi, (R+1)\sin\phi, 0)$ $\quad 0 \le \theta \le 2\pi$

FOR $\phi = \pi$ GET $((R-1)\cos\phi, (R-1)\sin\phi, 0)$

$S = \Phi(D)$ IS THE TORUS CENTERED AT $(0,0,0)$ WITH CENTRAL CIRCLE



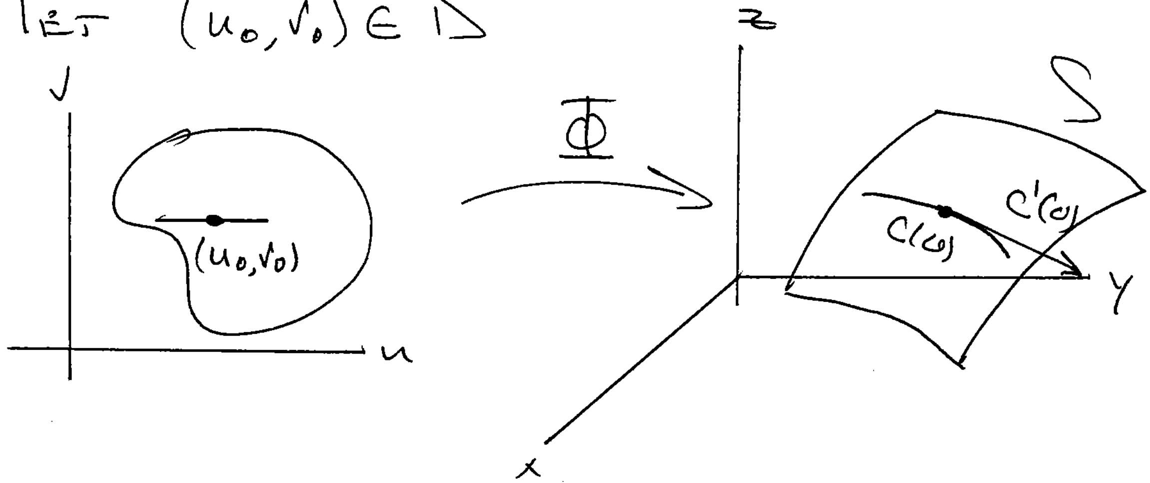
OF RADIUS R IN xy PLANE.

TANGENT PLANES TO PARAMETRIZED SURFACES

LET S BE PARAMETRIZED BY

$$\Phi : D \to \mathbb{R}^3$$

AND LET $(u_0, v_0) \in D$



FIX $\varepsilon > 0$ AND DEFINE $c : [-\varepsilon, \varepsilon] \to \mathbb{R}^3$ BY

$$c(t) = \Phi(u_0 + t, v_0).$$

THEN $c(t)$ IS A PARAMETRIZED CURVE IN S WITH $c(0) = \Phi(u_0, v_0)$ AND

$$c'(t) = \frac{\partial \Phi}{\partial u}(u_0 + t, v_0)$$

$$= \left(\frac{\partial x}{\partial u}(u_0+t, v_0), \frac{\partial y}{\partial u}(u_0+t, v_0), \frac{\partial z}{\partial u}(u_0+t, v_0) \right)$$

DEFINE THE VECTOR $T_u(u_0, v_0)$ TO BE

$$T_u(u_0,v_0) = c'(0) = \frac{\partial \Phi}{\partial u}(u_0,v_0)$$
$$= \left(\frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0)\right)$$

SIMILARLY WE DEFINE

$$T_v(u_0,v_0) = \frac{\partial \Phi}{\partial v}(u_0,v_0)$$
$$= \left(\frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0)\right)$$

BOTH T_u AND T_v ARE TANGENT TO THE SURFACE S AT $\Phi(u_0,v_0)$.

WE SAY S IS SMOOTH AT $\Phi(u_0,v_0)$ IF $T_u(u_0,v_0) \times T_v(u_0,v_0) \neq 0$. WE ALSO SAY $\Phi(u_0,v_0)$ IS A REGULAR POINT OF S.

(RECALL $T_u \times T_v \neq 0$ IFF T_u & T_v ARE NOT PARALLEL.)

S IS SAID TO BE SMOOTH OR REGULAR IF IT IS REGULAR AT ALL POINTS.

OBSERVE THAT $T_u \times T_v$ IS PERPENDICULAR TO THE PLANE SPANNED BY T_u & T_v, WHICH IS THE TANGENT PLANE TO S AT $\Phi(u_0,v_0)$. THE VECTOR

$$n = T_u \times T_v$$

IS SAID TO BE NORMAL TO S AT $\Phi(u_0,v_0)$.

IF $\Phi(u_0, v_0) = (x_0, y_0, z_0)$ THEN THE EQUATION OF THE TANGENT PLANE TO S AT (x_0, y_0, z_0) IS:

$$\boxed{(x - x_0, y - y_0, z - z_0) \cdot n = 0}$$

EX UNIT SPHERE ABOUT ORIGIN:

$$\Phi(\phi, \theta) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi) \quad \begin{cases} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$

FIND TANGENT PLANE AT $\Phi\left(\frac{\pi}{3}, \frac{\pi}{4}\right)$

$$= \left(\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}, \frac{1}{2}\right) = \left(\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, \frac{1}{2}\right)$$

$$T_\phi = (\cos\phi \cos\theta, \cos\phi \sin\theta, -\sin\phi)$$

$$T_\theta = (-\sin\phi \sin\theta, \sin\phi \cos\theta, 0)$$

$$T_\phi\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, -\frac{\sqrt{3}}{2}\right)$$

$$T_\theta\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \left(-\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, 0\right)$$

$$n\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \begin{vmatrix} i & j & k \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} & 0 \end{vmatrix} = \left(\frac{\sqrt{18}}{8}, \frac{\sqrt{18}}{8}, \frac{\sqrt{12}}{8}\right)$$

$$= \left(\frac{3\sqrt{2}}{8}, \frac{3\sqrt{2}}{8}, \frac{\sqrt{3}}{4}\right)$$

NOTE: $n = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, \frac{1}{2}\right)$ WHICH IS PARALLEL TO THE POSITION VECTOR $\Phi\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \left(\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, \frac{1}{2}\right)$, AS EXPECTED ON THE UNIT SPHERE ABOUT $(0,0,0)$.

THUS THE TANGENT PLANE HAS EQN.

$$\frac{\sqrt{3}}{2}\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, \frac{1}{2}\right) \cdot \left(x - \frac{\sqrt{6}}{4}, y - \frac{\sqrt{6}}{4}, z - \frac{1}{2}\right) = 0$$

$$\frac{\sqrt{6}}{4}\left(x - \frac{\sqrt{6}}{4}\right) + \frac{\sqrt{6}}{4}\left(y - \frac{\sqrt{6}}{4}\right) + \frac{1}{2}\left(z - \frac{1}{2}\right) = 0$$

$$\frac{\sqrt{6}}{4}x - \frac{6}{16} + \frac{\sqrt{6}}{4}y - \frac{6}{16} + \frac{1}{2}z - \frac{1}{4} = 0$$

$$\frac{\sqrt{6}}{4}x + \frac{\sqrt{6}}{4}y + \frac{1}{2}z = 1$$

$$\boxed{\sqrt{6}\,x + \sqrt{6}\,y + 2z = 4}$$

NOTE THIS IS EXACTLY WHAT WE GET IF WE TREAT S AS A LEVEL SURFACE OF $g(x,y,z) = x^2+y^2+z^2 = 1$. THEN $\nabla g = (2x, 2y, 2z) = 2(x,y,z)$ $\therefore$ $n = (x,y,z)$ SO AT $\left(\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, \frac{1}{2}\right)$ WE HAVE $n = \left(\frac{\sqrt{6}}{4}, \frac{\sqrt{6}}{4}, \frac{1}{2}\right)$ & SAME TANGENT PLANE.

EX IF S IS THE GRAPH $z = f(x,y)$ THEN IT HAS PARAMETRIZATION

$$\Phi(u,v) = (u, v, f(u,v))$$

SO

$$T_u = \left(1, 0, \frac{\partial f}{\partial u}\right) = \left(1, 0, \frac{\partial f}{\partial x}\right)$$

$$T_v = \left(0, 1, \frac{\partial f}{\partial v}\right) = \left(0, 1, \frac{\partial f}{\partial y}\right)$$

SO

$$n = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

SO EQN. OF TANGENT PLANE IS

$$(x - x_0, y - y_0, z - z_0) \cdot \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = 0$$

i.e. $$(x - x_0)\frac{\partial f}{\partial x} + (y - y_0)\cdot\frac{\partial f}{\partial y} - (z - z_0) = 0$$

i.e. $$\boxed{z = z_0 + (x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0)}$$

THIS EQN. WAS DERIVED IN AN EARLIER SECTION.