(7.2) Line Integrals

Defn

Let $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field which is cont. on a C^1 curve C, parametrized by

$$c: [a,b] \to \mathbb{R}^3$$

$$c(t) = (x(t), y(t), z(t))$$

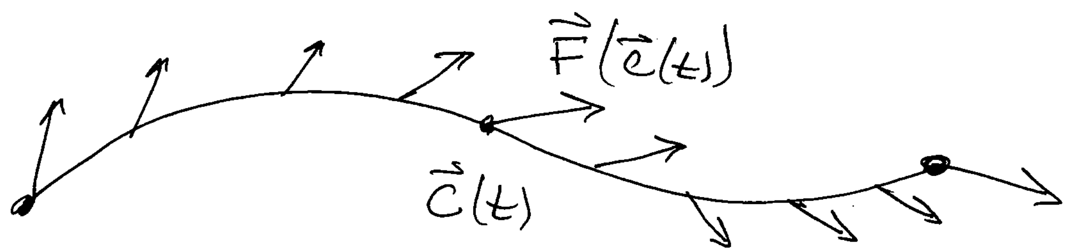
The <u>line integral</u> of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

$$\cancel{= \int_a^b \vec{F}(x(t), y(t), z(t))}$$

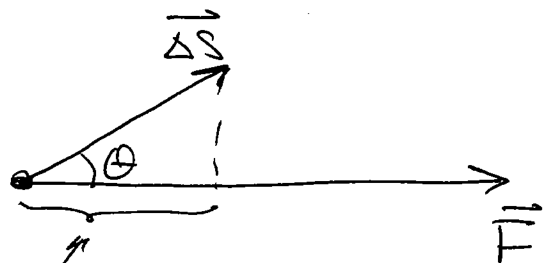
If $\vec{F}$ is only piecewise continuous, or C is only piecewise C^1, we simply sum over the pieces.

<u>Note</u>: Continuity of $\vec{F}$ guarantees existence of RHS.



$\int_C \vec{F} \cdot d\vec{s}$ HAS A NATURAL INTERPRETATION AS WORK.

IF A PARTICLE MOVES ALONG A DISPLACEMENT VECTOR $\overrightarrow{\Delta s}$ UNDER INFLUENCE OF A CONSTANT FORCE $\vec{F}$,



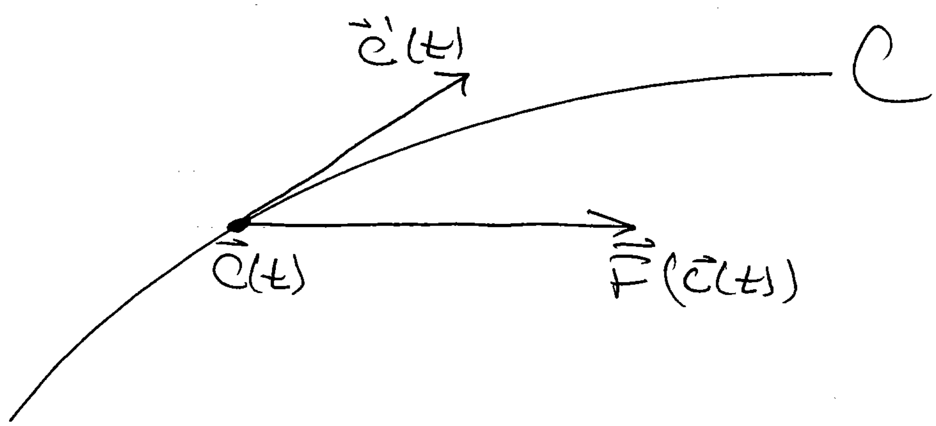
DISPLACEMENT IN DIR OF $\vec{F} = |\overrightarrow{\Delta s}| \cos\theta$

THEN THE WORK DONE BY $\vec{F}$ ON THE PARTICLE IS

$$\begin{aligned} \text{WORK} &= (\text{magnitude of } \vec{F})(\text{displacement in Dir } \vec{F}) \\ &= |\vec{F}|\,|\overrightarrow{\Delta s}| \cos\theta \\ &= \vec{F} \cdot \overrightarrow{\Delta s} \end{aligned}$$

IN THE MORE GENERAL SETTING, WHERE $\vec{F}$ CHANGES FROM POINT TO POINT AND DISPLACEMENT VECTOR CHANGES FROM POINT TO POINT, THE INFINITESIMAL WORK IS

$$dW = \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$



AND THE TOTAL WORK IS

$$\text{WORK} = \int_C dW = \int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

THIS INTERPRETATION CAN BE MADE PRECISE BY ~~DISSECTING~~ APPROXIMATING C INTO POLYGONAL SEGMENTS AND FORMING A LIMIT OF RIEMANN SUMS.

FORMALLY

$$d\vec{s} = \vec{c}\,'(t)\, dt = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt$$
$$= (dx, dy, dt).$$

SO IF $\vec{F} = (\vec{F}_1, \vec{F}_2, \vec{F}_3)$

THE LINE INTEGRAL IS OFTEN WRITTEN AS

$$\int_C \vec{F}\cdot d\vec{s} = \int_C \underbrace{F_1\,dx + F_2\,dy + F_3\,dz}_{\text{DIFFERENTIAL FORM}}$$

(EVERYTHING WE'VE DONE GOES ALSO FOR 2-DIMENSIONS.)

Ex $\vec{F}(x,y,z) = (x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{c}(t) = (t, t, t^2) \qquad 0 \le t \le 1$

$$\int_C \vec{F}\cdot d\vec{s} = \int_0^1 (t,t,t^2)\cdot(1,1,2t)\,dt$$

$$= \int_0^1 t + t + 2t^3\,dt = 2\int_0^1 (t + t^3)\,dt$$

$$= 2\left(\tfrac{1}{2}t^2 + \tfrac{1}{4}t^3\right)\Big|_0^1 = 2\left(\tfrac{1}{2} + \tfrac{1}{4}\right) = 2\cdot\tfrac{3}{4} = \boxed{\tfrac{3}{2}}$$

~~Ex~~ $\cancel{C: \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \qquad 0 \le t \le 2\pi}$

$$\cancel{\int_C yz\,dx + xz\,dy + xy\,dz = \int_0^{2\pi} -t\sin t\cos t}$$

Ex.

$$\int_C \cos z \, dx + e^x dy + e^y dz$$

$$C: \begin{cases} x = 1 \\ y = t \\ z = e^t \end{cases} \qquad 0 \le t \le 2$$

$$= \int_0^2 \left(\cos(e^t) \cdot 0 + e \cdot 1 + e^t \cdot e^t \right) dt$$

$$= \int_0^2 \left(e + e^{2t} \right) dt = et + \frac{1}{2} e^{2t} \Big|_0^2$$

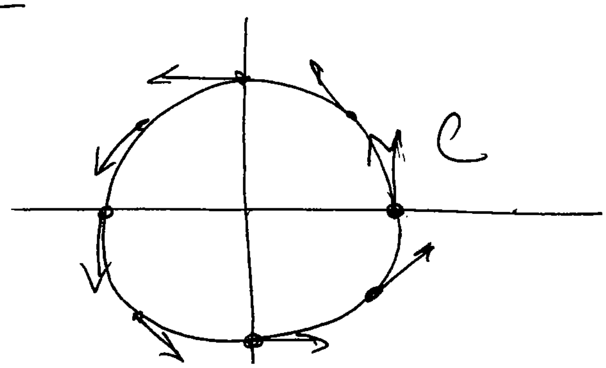
$$= \boxed{2e + \frac{e^4}{2} - \frac{1}{2}}$$

~~Ex.~~

Ex $C \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \le t \le 2\pi$

$$\int_C -y\,dx + x\,dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \boxed{2\pi}$$

THIS IS THE WORK BY $\vec{F} = (-y, x)$ ON A PARTICLE TRAVERSING THE UNIT CIRCLE



NOTE: $\vec{F}$ HAPPENS TO BE EVERYWHERE TANGENT TO C

OBSERVE THAT THE line INTEGRAL DEPENDS ON ORIENTATION OF C.

$$-C : \begin{cases} x = \cos(-t) = \cos t \\ y = \sin(-t) = -\sin t \end{cases}$$

$$\int_{-C} -y\,dx + x\,dy = \int_0^{2\pi} -(\sin^2 t - \cos^2 t)\,dt = -2\pi .$$

NEGATIVE WORK INDICATES THAT $\vec{F}$ IMPEDES MOTION.

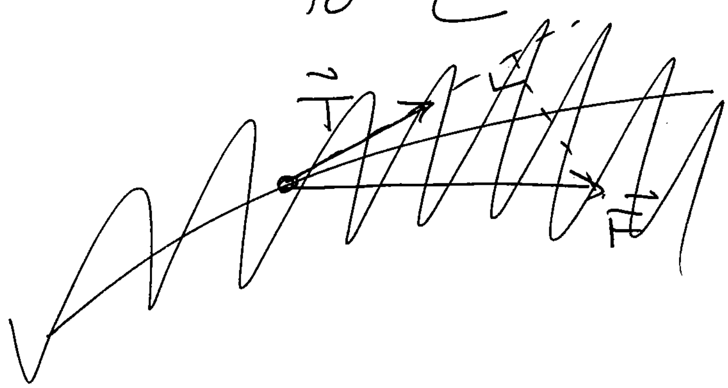
OBSERVE THAT

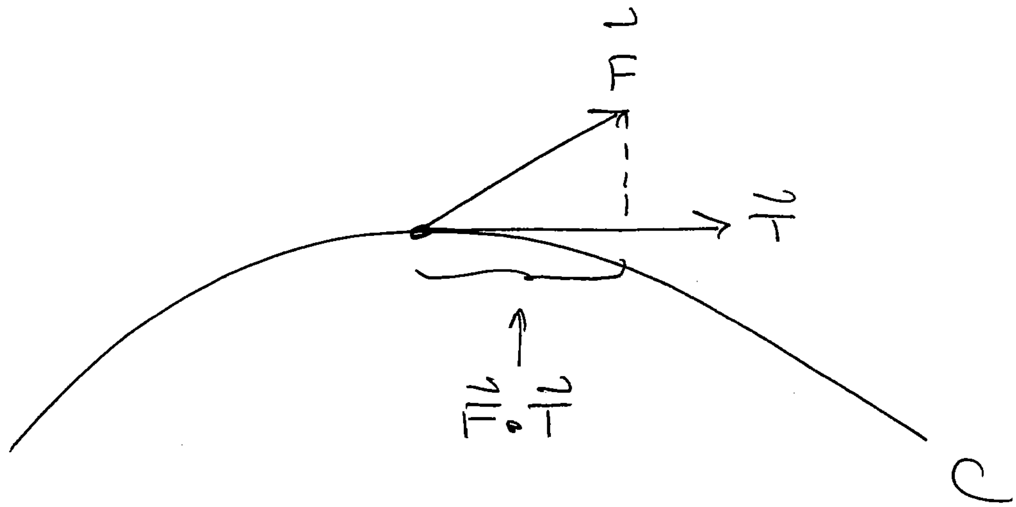
$$\vec{F}(\vec{c}(t))\cdot\vec{c}\,'(t) = \left(\vec{F}(c(t))\cdot\frac{c'(t)}{|c'(t)|}\right)|c'(t)|$$

$$= \vec{F}(c(t))\cdot\underbrace{\vec{T}(c(t))}_{\text{UNIT TANGENT TO } C}\,|c'(t)|$$

$$\therefore \int_C \vec{F}\cdot d\vec{s} = \int_a^b \left(\vec{F}(c(t))\cdot\vec{T}(\vec{c}(t))\right)|c'(t)|\,dt$$

$$= \int_C \underbrace{(\vec{F}\cdot\vec{T})}_{\substack{\text{COMPONENT OF } F \text{ WHICH}\\ \text{IS IN DIRECTION TANGENT}\\ \text{TO } C}}\,ds$$





IN THE LAST EXAMPLE, SINCE $\vec{F} = \vec{T}$ ON C WE HAVE $\vec{F}\cdot\vec{T} = \vec{T}\cdot\vec{T} = 1$ SO

$$\int_C \vec{F}\cdot d\vec{s} = \int_C (\vec{F}\cdot\vec{T})\, ds = \int_C ds = 2\pi$$

LET $h : [a_1, b_1] \longrightarrow [a_2, b_2]$ BE A BIJECTION WHICH IS C^1 AND HAS C^1 INVERSE h^{-1} (SUCH A MAP IS CALLED A <u>DIFFEOMORPHISM</u>).

THEN $c_2 : [a_2, b_2] \longrightarrow \mathbb{R}^3$ AND $c_1 : [a_1, b_1] \longrightarrow \mathbb{R}^3$ WHERE

$$c_1(t) = c_2(h(t))$$

ARE TWO DIFFERENT PARAMETRIZATIONS OF THE SAME (ORIENTED) CURVE C.

Thm Let $\vec{F}$ be cont. on C. Then

$$\int_{a_1}^{b_1} \vec{F}(c_1(t)) \cdot c_1'(t)\,dt = \int_{a_2}^{b_2} \vec{F}(c_2(t)) \cdot c_2'(t)\,dt$$

Proof: We have

$$c_1'(t) = \frac{d}{dt}\, c_2(h(t))$$

$$= c_2'(h(t))\, h'(t)$$

Thus

$$\text{LHS} = \int_{a_1}^{b_1} \vec{F}(c_2(h(t))) \cdot c_2'(h(t))\, h'(t)\,dt$$

Let $u = h(t)$
$du = h'(t)\,dt$

$a_1 \le t \le b_1 \iff a_2 \le u \le b_2$

$$= \int_{a_2}^{b_2} \vec{F}(c_2(u)) \cdot c_2'(u)\,du$$

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SPECIAL CASE: $\vec{F}$ IS A GRADIENT FIELD.

SUPPOSE $f : \mathbb{R}^3 \to \mathbb{R}$ IS C^1 AND
$\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$.

THEN BY THE CHAIN RULE

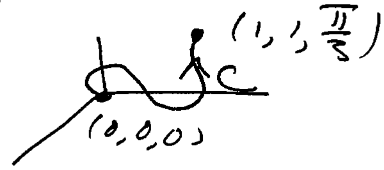
$$\frac{d}{dt} f(c(t)) = \nabla f(c(t)) \cdot c'(t)$$

$$\therefore \int_C \vec{F} \cdot d\vec{s} = \int_a^b \nabla f(c(t)) \cdot c'(t)\, dt$$

$$= \int_a^b \frac{d}{dt}\left(f(c(t))\right) dt$$

$$= f(c(b)) - f(c(a))$$

EX. $\int_C (4x-y)\,dx - x\,dy - \sin z\,dz$



$\vec{F} = (4x-y, -x, -\sin z) = \nabla f$, $f(x,y,z) = 2x^2 - xy + \cos z$

$= f(1,1,\frac{\pi}{3}) - f(0,0,0) = \boxed{\frac{1}{2}}$.