(7.1) PATH INTEGRALS

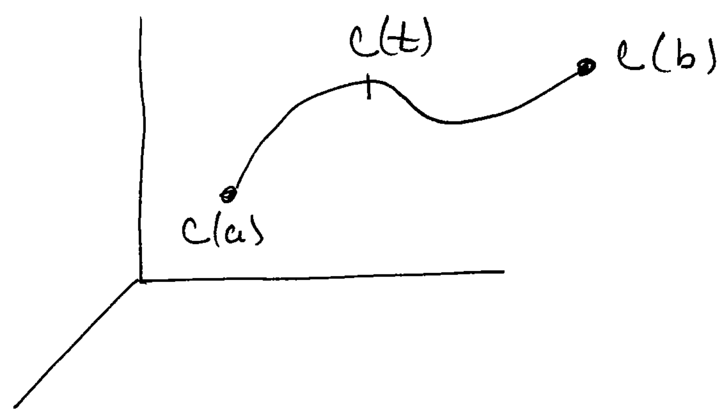
LET $f : \mathbb{R}^3 \to \mathbb{R}$, i.e. f ASSIGNS A NUMBER TO EACH POINT (x, y, z) IN SPACE. WE CAN INTERPRET f AS SOMETHING LIKE TEMPERATURE, OR MASS DENSITY OF SOME SOLID OCCUPYING $\mathbb{R}^3$.

LET C BE A C^1 (i.e. CONTINUOUSLY DIFFERENTIABLE) CURVE IN SPACE, i.e. C IS THE IMAGE OF A MAP

$$\vec{c} : [a, b] \to \mathbb{R}^3$$

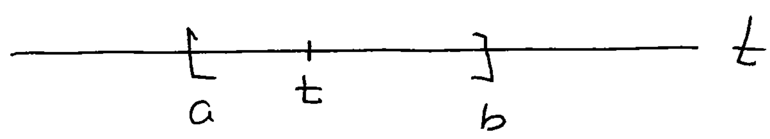
$$\vec{c}(t) = (x(t), y(t), z(t))$$

WHERE $x(t), y(t), z(t)$ ARE REAL VALUED FUNCTIONS ON $[a, b]$ WHICH HAVE CONTINUOUS ~~DERIVATES~~ DERIVATIVES.



PARAMETRIC EQUATIONS FOR C ARE:

$$C : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$



DEFN:
THE INTEGRAL OF f ALONG C is THE NUMBER

$$\int_C f\,ds = \int_a^b f(c(t))\cdot|c'(t)|\,dt$$

$$= \int_a^b f(x(t),y(t),z(t))\cdot\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\;dt$$

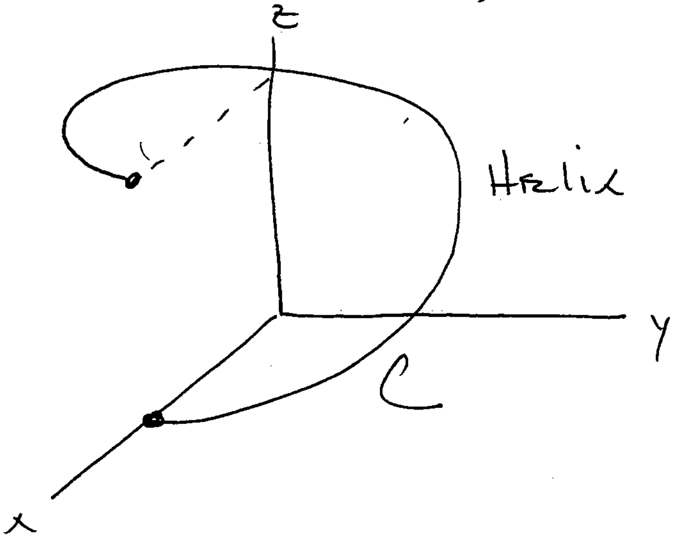
IF C is only PIECEWISE CONTINUOUS WE DEFINE $\int_C f\,ds$ BY BREAKING C INTO C^1 PIECES AND ADDING THE RESULTING INTEGRALS.

WE SAY f is INTEGRABLE ALONG C IF THE ABOVE INTEGRAL EXISTS.

OBSERVE: APPARENTLY $\int_C f\,ds$ DEPENDS ON THE PARTICULAR PARAMETRIZATION OF C. WE WILL SEE IN SUBSEQUENT SECTIONS THAT IT DOES NOT, i.e. $\int_C f\,ds$ DEPENDS ONLY ON THE FUNCTION f AND THE CURVE (i.e. THE SET) C.

$f(x,y,z) = x^2+y^2+z^2$

Ex. $c(t) = (\cos t, \sin t, t)$ for $0 \le t \le 2\pi$



$c'(t) = (-\sin t, \cos t, 1)$

$|c'(t)| = \sqrt{\cos^2 t + \sin^2 t + 1^2} = \sqrt{2}$

$f(c(t)) = \cos^2 t + \sin^2 t + t^2 = 1 + t^2$

$$\int_C f\,ds = \int_0^{2\pi} (1+t^2)\cdot\sqrt{2}\,dt = \dots = \frac{2\sqrt{2}\,\pi}{3}(3+4\pi^2)$$

Ex. SAME f. $d(t) = (\cos 2t, \sin 2t, 2t)$ $0 \le t \le \pi$

SO DIFF. PARAM. OF SAME CURVE C.

GET SAME VALUE!

$$d'(t) = (-2\sin 2t,\ 2\cos 2t,\ 2)$$

$$|d'(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 4} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$f(d(t)) = \cos^2 2t + \sin^2 2t + 4t^2 = 1+4t^2$$

$$\int_C f\,ds = \int_0^{\pi} (1+4t^2)\,2\sqrt{2}\,dt = 2\sqrt{2}\left(t + \frac{4}{3}t^3\right)\Big|_0^{\pi}$$

$$= 2\sqrt{2}\left(\pi + \frac{4}{3}\pi^3\right) = \frac{2\sqrt{2}\,\pi}{3}\left(3+4\pi^2\right).$$

NOTE WHEN $f(x,y,z) = 1$, THEN $\int_C f\,ds$ IS JUST THE LENGTH OF C (WHICH WE EXPECT IS INDEPENDENT OF PARAMETRIZATION)

<u>Ex</u> $c(t) = (\cos t, \sin t, t)$ $\quad 0 \le t \le 2\pi$

$f(x,y,z) = 1$.

$$\text{length}(C) = \int_C ds = \int_0^{2\pi} |c'(t)|\,dt = \sqrt{2}\int_0^{2\pi} dt$$

$$= \sqrt{2}\cdot 2\pi = \boxed{2\sqrt{2}\,\pi}$$

IF $f(x,y,z)$ REPRESENTS THE MASS DENSITY ALONG C THEN

$$\int_C f\,ds$$

WOULD BE THE <u>TOTAL MASS</u>. IF $f(x,y,z)$ REPRESENTS TEMPERATURE ALONG C THEN THE ABOVE PATH INTEGRAL WOULD BE THE <u>TOTAL HEAT ENERGY</u> AND

$$\frac{\int_C f\,ds}{\text{length}(C)}$$

WOULD BE THE <u>AVERAGE TEMPERATURE</u> ALONG C.

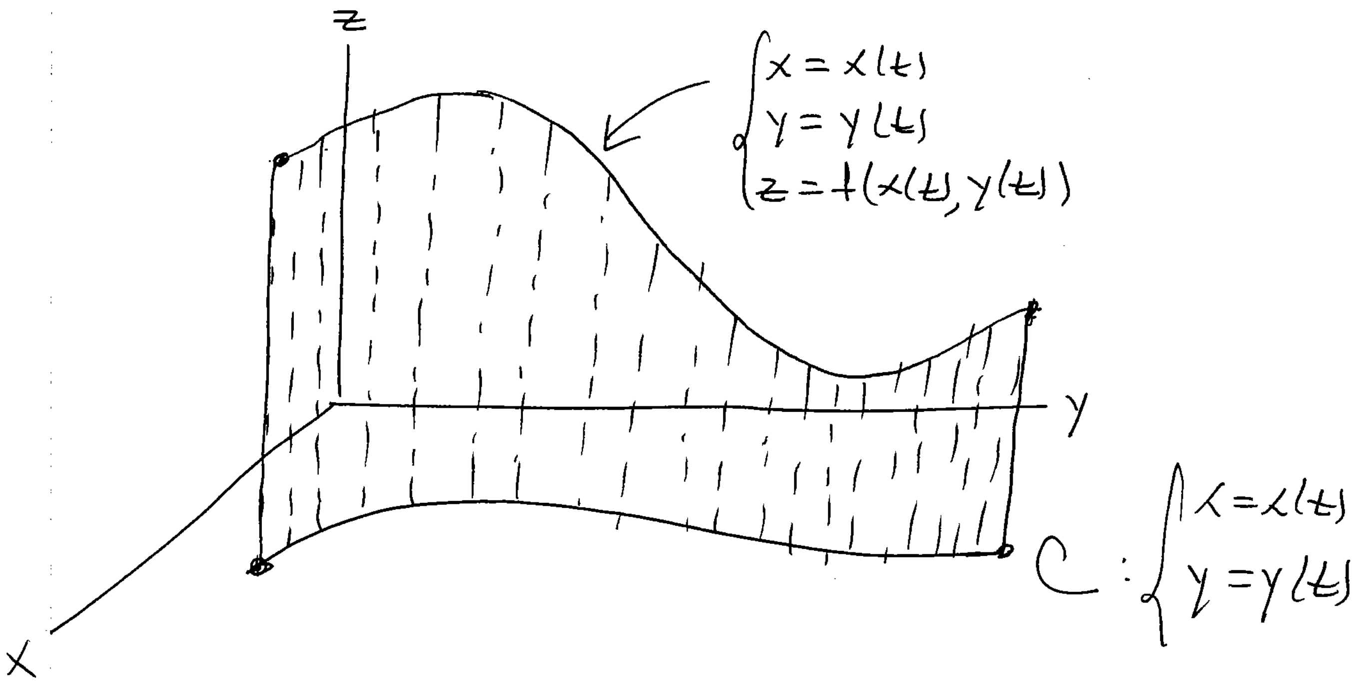
<u>EX</u>. SAME AS ABOVE.

$$\text{AVG. VALUE OF } f \text{ ALONG } C = \frac{\frac{2\sqrt{2}\pi}{3}(3+4\pi^2)}{2\sqrt{2}\pi}$$

$$= \boxed{1+\frac{4}{3}\pi^2}.$$

All THESE SAME DEFINITIONS AND INTERPRETATIONS CAN BE MADE FOR 2-dim CURVES TOO.

HOWEVER, IN 2-dim WE HAVE AN ADDED OBVIOUS INTERPRETATION.



$\int_C f\,ds$ = AREA OF VERTICAL SURFACE UNDER THE CURVE $(x(t), y(t), f(x(t), y(t)))$.

HW (7.1): 2a, 3a, 4ab, 5, 8c, 9, 11