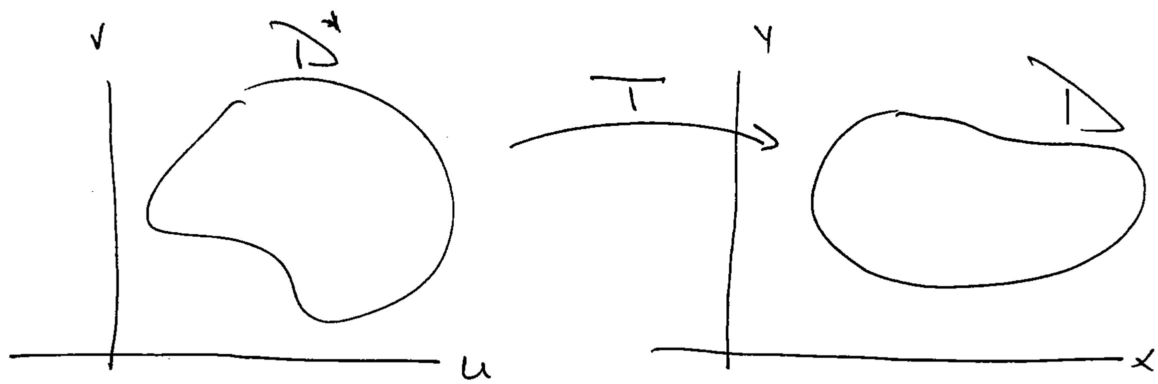
(6.2) CHANGE OF VARIABLES THM

OUR GOAL IN THIS SECTION IS TO SHOW HOW TO CHANGE VARIABLE IN DOUBLE AND TRIPLE INTEGRALS.



$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} ? \quad du\,dv$$

SAY $T: D^* \to D$ IS GIVEN BY $T(u,v) = (x(u,v), y(u,v))$ ONE MIGHT GUESS

$$? = f(x(u,v), y(u,v)).$$

IF THIS WERE CORRECT THEN FOR $f = 1$ WE WOULD GET

$$A(D) = \iint_D \cancel{f}\, dA = \iint_{D^*} dA = A(D^*).$$

SINCE THIS IS IN GENERAL FALSE, ? ABOVE IS INCORRECT.

INSTEAD ? MUST CONTAIN A FACTOR WHICH ACCOUNTS FOR THE WAY T WARPS AREA IN D^* INTO AREA IN D.

RECALL THAT THE DERIVATIVE OF $T(u,v) = (x(u,v), y(u,v))$ IS THE MATRIX

$$DT(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

<u>DEFN</u>:
THE <u>JACOBIAN DETERMINANT</u> (OR JUST <u>JACOBIAN</u>) OF T IS THE DETERMINANT OF $DT(u,v)$, i.e.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\big(DT(u,v)\big)$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

<u>EX</u>. $T: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

Ex. $T : \begin{cases} x = u + v \\ y = u - v \end{cases}$ $\qquad DT(u,v) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

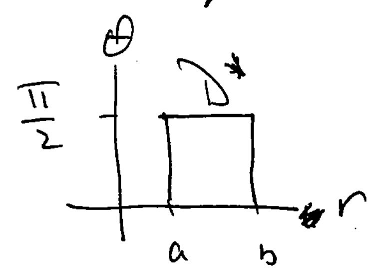
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

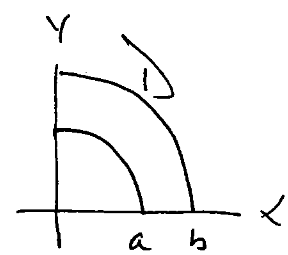
WE WISH TO SHOW THAT IN GENERAL (i.e. EVEN FOR nonlinear MAPS) THE JACOBIAN SAYS PRECISELY HOW T WARPS AREA AT EACH POINT.

IN PARTICULAR WE SHOW:

$$A(D) = \iint_D dx\,dy = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv .$$

Ex. $T : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$





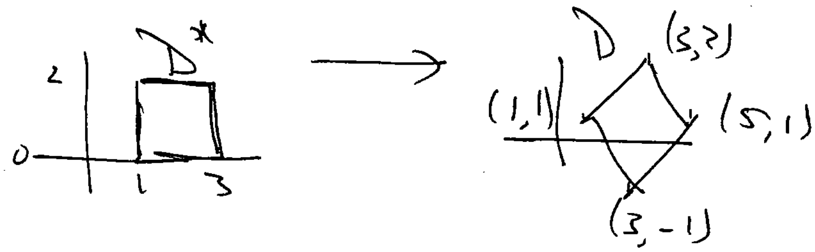
$$A(D) = \iint_{D^*} r\,dr\,d\theta = \int_0^{\pi/2}\int_a^b r\,dr\,d\theta = \int_0^{\pi/2} \tfrac{1}{2}(b^2 - a^2)\,d\theta$$

$$= \frac{\pi}{4}(b^2 - a^2) .$$

CHANGE OF VAR THM:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v),\, y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

Ex



$$\begin{cases} x = u+v \\ y = u-v \end{cases}$$

$$\iint_D e^{x+y}\,dx\,dy = \iint_{D^*} 2e^{2u}\,du\,dv$$

$\vdots$