(6.1) GEOMETRY OF $\mathbb{R}^2 \to \mathbb{R}^2$ MAPS.

LET $D^* \subseteq S^* \subseteq \mathbb{R}^2$ AND LET $T: S^* \to \mathbb{R}^2$ BE CONTINUOUSLY DIFFERENTIABLE. THIS MEANS THAT IF $T(u,v) = (x(u,v), y(u,v))$ THEN

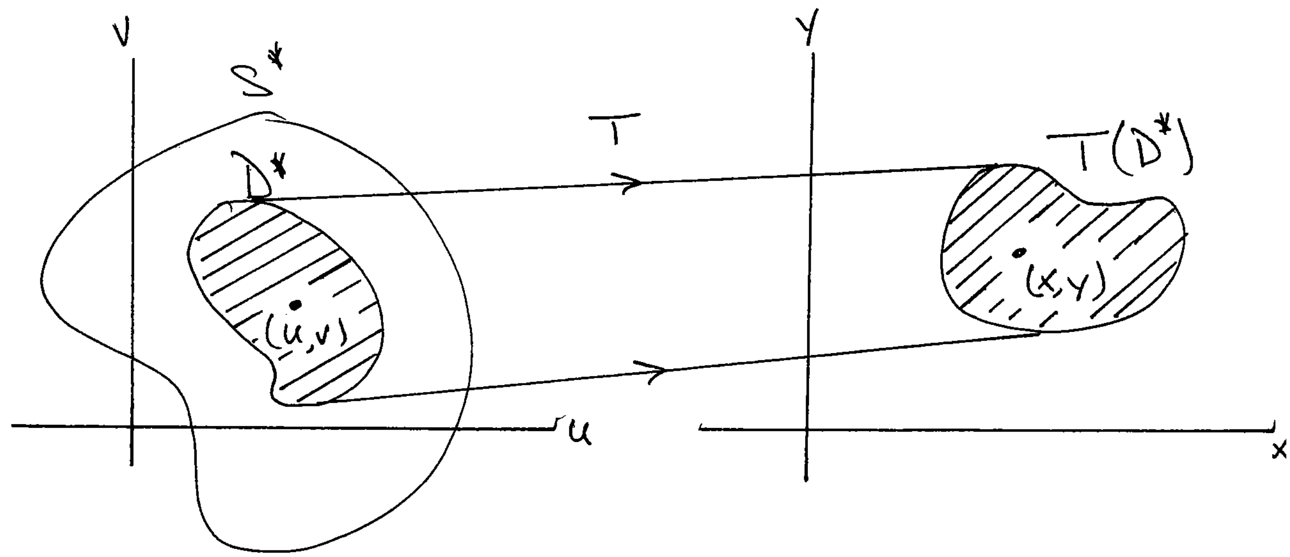
$$\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$$

EXIST AND ARE CONTINUOUS ON S^*.

DEFN
THE IMAGE OF D^* UNDER T IS THE SET

$$T(D^*) = \{ T(u,v) \mid (u,v) \in D^* \}$$

i.e. $(x,y) \in T(D^*)$ IFF THERE EXISTS $(u,v) \in D^*$ SUCH THAT $T(u,v) = (x,y)$.

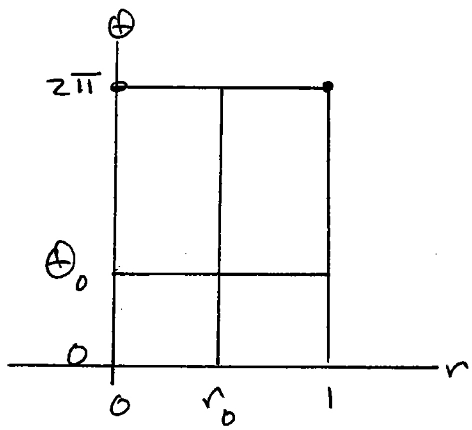


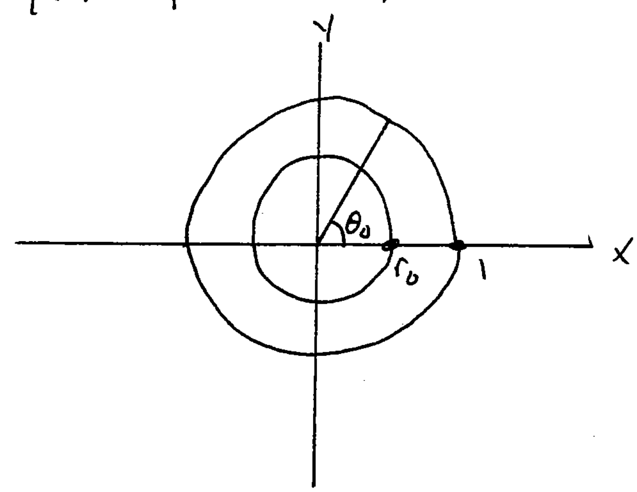
THE IMAGE OF THE DOMAIN S^* ITSELF UNDER T IS CALLED THE IMAGE OF T: $\text{image}(T) = T(S^*)$
OFTEN, BUT NOT ALWAYS, S^* IS THE WHOLE PLANE $\mathbb{R}^2$.

EX. LET $T:\mathbb{R}^2 \to \mathbb{R}^2$ BE THE POLAR COORDINATE TRANSFORMATION $T(r,\theta) = (r\cos\theta, r\sin\theta)$, i.e.

$$T: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

AND LET $D^* = [0,1]\times[0,2\pi] = \{(r,\theta) \mid 0 \le r \le 1 \;\&\; 0 \le \theta \le 2\pi\}$.





THUS $T(D^*) = \{(x,y) \mid x^2+y^2 \le 1\}$ WHICH IS THE UNIT DISC CENTERED AT THE ORIGIN IN THE xy PLANE.

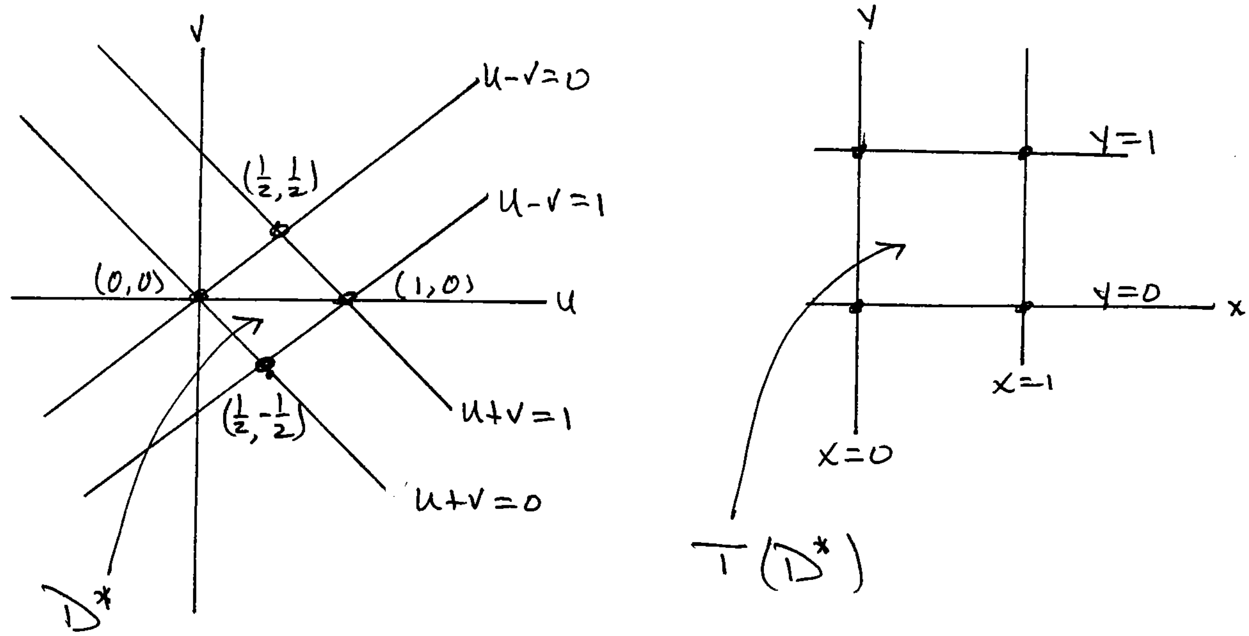
EX.
LET $T: D^* \to \mathbb{R}^2$ BE GIVEN BY

$$T(u,v) = (u+v, u-v)$$

WHERE D^* IS THE REGION IN THE uv-PLANE BOUNDED BY THE LINES $u+v=0$, $u-v=0$, $u+v=1$, AND $u-v=1$. THUS

$$T: \begin{cases} x = u+v \\ y = u-v \end{cases} \quad \text{FOR } (u,v) \in D^*$$

WE SEE THAT $T(D^*)$ IS THE REGION IN THE xy-PLANE BOUNDED BY THE LINES $x=0$, $x=1$, $y=0$, AND $y=1$.



DEFN

A MAP $T:\mathbb{R}^2 \to \mathbb{R}^2$ IS CALLED <u>LINEAR</u> IFF THERE ARE CONSTANTS a, b, c, d SUCH THAT

$$T(u,v) = (au+bv,\ cu+dv)$$

SUCH A MAP IS SPECIFIED WHEN WE GIVE A 2×2 MATRIX

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

THEN $T(u,v) = A\cdot\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au+bv \\ cu+dv \end{pmatrix}.$

THE PRECEDING EXAMPLE WAS A LINEAR MAP WITH $a=b=c=1$ AND $d=-1$.

THEOREM

IF $T : \mathbb{R}^2 \to \mathbb{R}^2$ IS LINEAR, THEN T TRANSFORMS LINES INTO LINES, AND HENCE POLYGONS INTO POLYGONS.

PROOF:

LET T BE GIVEN BY

$$T : \begin{cases} x = au + bv \\ y = cu + dv \end{cases}$$

AND CONSIDER THE LINE $v = \alpha u + \beta$ IN THE uv PLANE. APPLYING T TO A POINT (u, v) ON THIS LINE GIVES xy COORDINATES

$$x = au + b(\alpha u + \beta) = (a + b\alpha)u + b\beta$$
$$y = cu + d(\alpha u + \beta) = (c + d\alpha)u + d\beta$$

THUS $u = \dfrac{x - b\beta}{a + b\alpha}$, AND HENCE

$$y = (c + d\alpha)\left(\frac{x - b\beta}{a + b\alpha}\right) + d\beta$$

i.e.

$$y = \left(\frac{c + d\alpha}{a + b\alpha}\right)x + \left(d\beta - b\beta\left(\frac{c + d\alpha}{a + b\alpha}\right)\right)$$

WHICH IS THE EQUATION OF A LINE IN THE xy-PLANE.

///

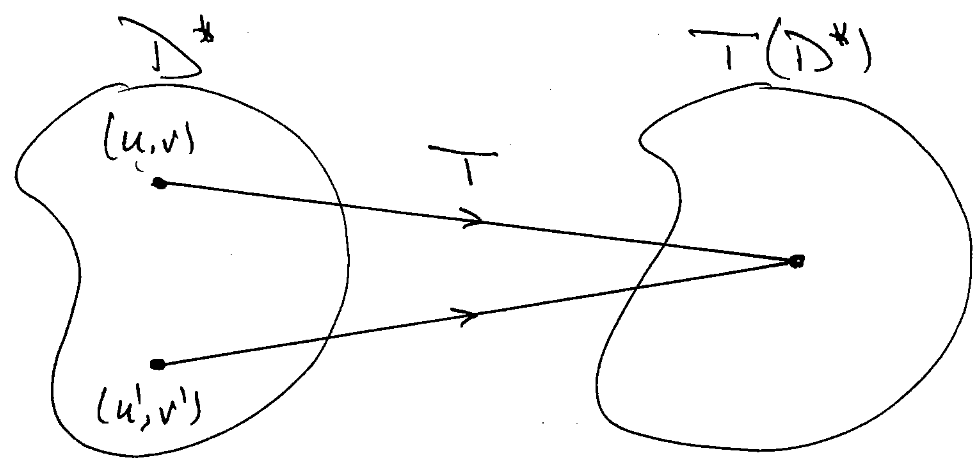
DEFN

A MAP $T: D^* \to \mathbb{R}^2$ IS SAID TO BE ONE-TO-ONE ON D^* (OR INJECTIVE ON D^*) IFF FOR ANY POINTS $(u,v) \in D^*$ AND $(u',v') \in D^*$ WE HAVE

$$T(u,v) = T(u',v') \quad \text{implies} \quad (u,v) = (u',v').$$

i.e. T IS ONE-TO-ONE ON D^* IF NO TWO DISTINCT POINTS ARE MAPPED TO ONE AND THE SAME POINT IN $T(D^*)$.

THE FOLLOWING PICTURE SHOWS WHAT DOES NOT HAPPEN IN A ONE-TO-ONE MAP!



EX. THE POLAR COORDINATE MAP

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

- IS NOT ONE-TO-ONE ON $[0,1] \times [0, 2\pi]$
- IS ONE-TO-ONE ON $(0,1] \times [0, 2\pi)$

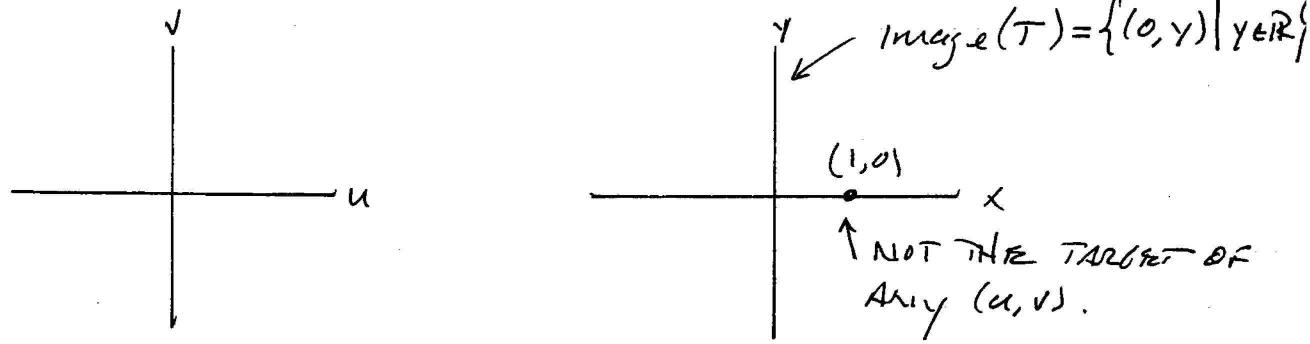
DEFN

A MAP $T : D^* \to \mathbb{R}^2$ IS SAID TO BE ONTO A SET $D \subseteq \mathbb{R}^2$ (ALSO SURJECTIVE) IFF FOR ALL POINTS $(x,y) \in D$, THERE EXISTS $(u,v) \in D^*$ SUCH THAT

$$T(u,v) = (x,y),$$

i.e. T is ONTO D IFF $D \subseteq \text{image}(T)$. IN PARTICULAR, EVERY MAP IS ONTO ITS OWN IMAGE.

EX $T : \mathbb{R}^2 \to \mathbb{R}^2$ GIVEN BY $T(u,v) = (0,v)$



- T IS NOT ONTO $\mathbb{R}^2$ SINCE THERE DOES NOT EXIST $(u,v) \in \mathbb{R}^2$ SUCH THAT $T(u,v) = (1,0)$.

- T IS ONTO ANY SUBSET OF THE y-AXIS, SINCE $T(u,y) = (0,y)$.

DEFN.

A MAP $T : D^* \to D$ WHICH IS BOTH ONE-TO-ONE ON D^* AND ONTO D IS CALLED A BIJECTION, ALSO A ONE-TO-ONE CORRESPONDANCE.

DEFN.

THE DETERMINANT OF A 2×2 MATRIX

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

IS THE NUMBER

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

IF $T: \mathbb{R}^2 \to \mathbb{R}^2$ IS THE LINEAR MAP CORRESPONDING TO MULTIPLICATION BY A, i.e.

$$T(u,v) = (au+bv,\ cu+dv),$$

THEN THE DETERMINANT OF T IS

$$\det(T) = \det(A) = ad - bc$$

THEOREM

A LINEAR MAP T IS ONE-TO-ONE AND ONTO IFF $\det(T) \neq 0$.

PROOF: (THAT $\det(T) \neq 0 \Rightarrow$ ONE-TO-ONE)

ASSUME $T(u,v) = (au+bv,\ cu+dv)$ AND $ad-bc \neq 0$. SUPPOSE

$$T(u,v) = T(u',v')$$

FOR SOME POINTS $(u,v), (u',v') \in \mathbb{R}^2$. WE

MUST SHOW $(u,v) = (u',v')$, i.e. $u = u'$ AND $v = v'$. WE HAVE

$$\begin{cases} au + bv = au' + bv' \\ cu + dv = cu' + dv' \end{cases}$$

$$\therefore \begin{cases} adu + bdv = adu' + bdv' \\ -bcu - bdv = -bcu' - bdv' \end{cases}$$

$$\therefore \quad (ad-bc)u = (ad-bc)u'$$

$$\therefore \quad u = u'$$

THUS
$$\begin{cases} bv = bv' \\ dv = dv' \end{cases}$$

NOW $ad - bc \neq 0$ IMPLIES THAT NOT BOTH $b=0$ AND $d=0$, HENCE EITHER $b \neq 0$ OR $d \neq 0$. IN EITHER CASE WE GET

$$v = v'$$

AS REQUIRED. ///.

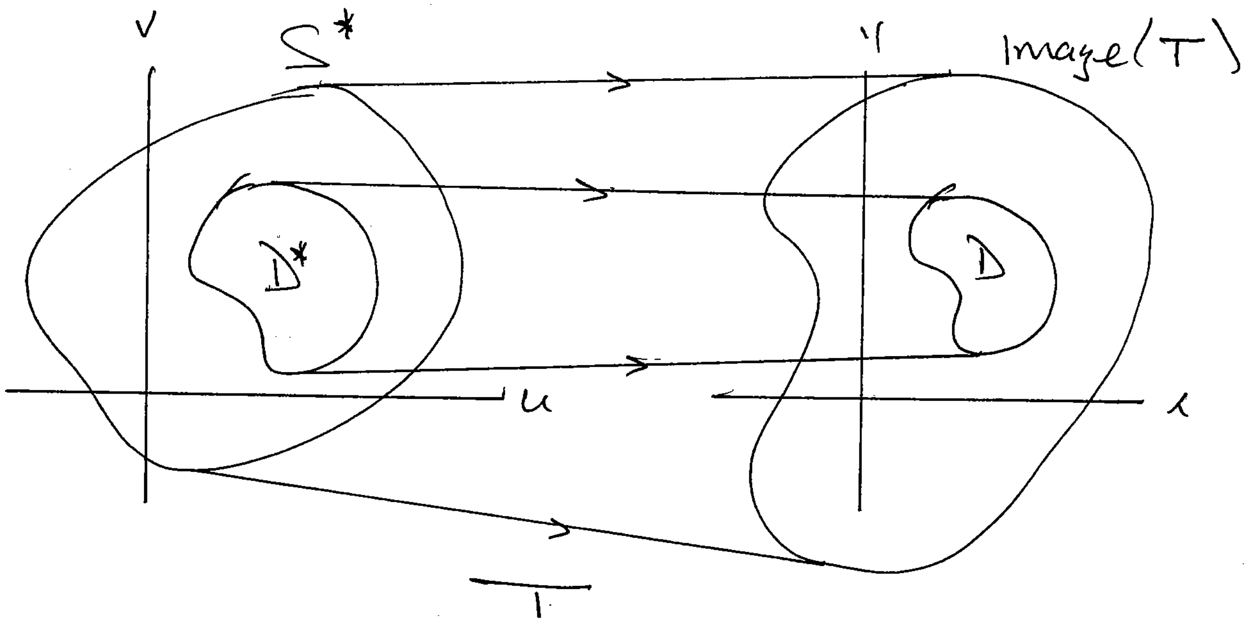
<u>EXERCISE</u>: SHOW T ONE-TO-ONE $\Rightarrow \det(T) \neq 0$. (THIS IS (6.1) #8.)

<u>EXERCISE</u>: SHOW T IS ONTO $\Longleftrightarrow \det(T) \neq 0$. (THIS IS (6.1) #9.)

CONSIDER THE FOLLOWING PROBLEM:

GIVEN $T: S^* \to \mathbb{R}^2$ WHICH IS ONE-TO-ONE, AND GIVEN A REGION $D \subseteq \text{Image}(T)$, FIND A REGION $D^* \subseteq S^*$ SUCH THAT

$$T(D^*) = D$$



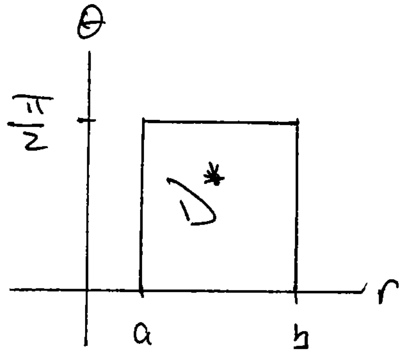
WE CALL D^* THE <u>PRE-IMAGE OF D UNDER T</u> OR THE <u>PULL-BACK OF D BY T</u> AND WRITE

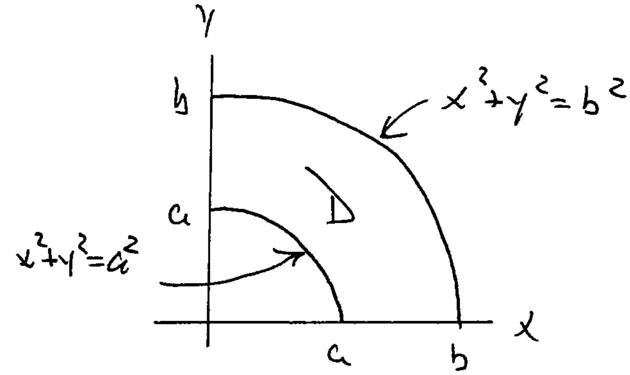
$$D^* = T^{-1}(D).$$

<u>EX.</u> LET $T: \mathbb{R}^2 \to \mathbb{R}^2$ BE GIVEN BY $T(r,\theta) = (r\cos\theta, r\sin\theta)$ AND

$$D = \{(x,y) \mid x \geq 0, y \geq 0, a^2 \leq x^2 + y^2 \leq b^2\}.$$

WHERE $0 < a < b$.





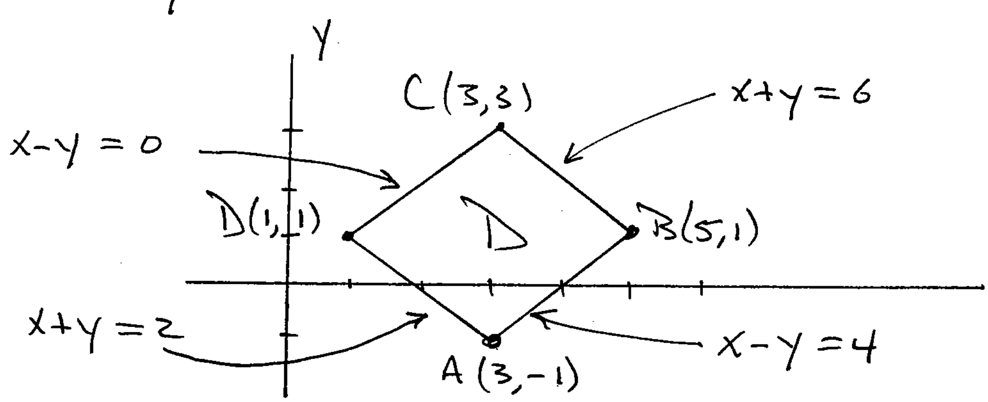
THEN $D^* = T^{-1}(D) = \{(r,\theta) \mid a \le r \le b,\ 0 \le \theta \le \frac{\pi}{2}\}$.

NOTE! AS WAS MENTIONED PREVIOUSLY, T IS NOT ONE-TO-ONE ON $\mathbb{R}^2$, BUT IT IS ONE-TO-ONE ON A SUBSET $S^* \subseteq \mathbb{R}^2$ WHICH CONTAINS D^*. FOR INSTANCE WE CAN TAKE S^* TO BE THE UPPER HALF-PLANE:

$$D^* \subseteq S^* = \{(r,\theta) \mid r > 0,\ 0 \le \theta \le \pi\}$$

EX. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(u,v) = (u+v,\ u-v)$.
THEN
$$\det(T) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0$$

WHENCE T IS A BIJECTION. LET D BE THE REGION IN THE xy PLANE BOUNDED BY THE LINES $x-y=0$, $x-y=4$, $x+y=6$, AND $x+y=2$.



$\boxed{\text{AREA} = 8}$

WE SOLVE THE EQUATIONS FOR T

$$T: \begin{cases} x = u + v \\ y = u - v \end{cases}$$

TO GET

$$T^{-1}: \begin{cases} u = \frac{1}{2}(x+y) \\ v = \frac{1}{2}(x-y) \end{cases}$$

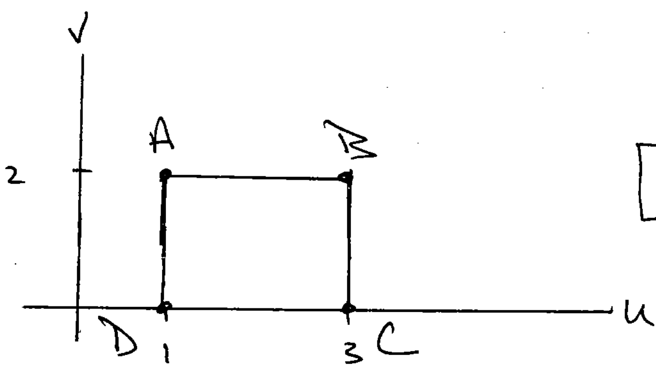
THEN PULLING BACK THE POINTS A, B, C, D

$$(1,2) \xleftarrow{T^{-1}} A(3,-1)$$

$$(3,2) \longleftarrow B(5,1)$$

$$(3,0) \longleftarrow C(3,3)$$

$$(1,0) \longleftarrow D(1,1)$$



$\boxed{\text{AREA} = 4}$

OBSERVE $\det(T) = -2 = (-1)(2)$

CHANGE IN ORIENTATION

CHANGE IN AREA