(5.5) THE TRIPLE INTEGRAL

LET $B = [a,b] \times [c,d] \times [p,q]$ BE A 3-dim. RECTANGLE IN $\mathbb{R}^3$ WHOSE SIDES ARE PARALLEL TO THE COORDINATE PLANES.

A <u>REGULAR PARTITION</u> OF B CONSISTS OF 3 SETS OF EQUALLY SPACED POINTS $\{x_i\}, \{y_j\}, \{z_k\}$, WHERE

$$a = x_0 < x_1 < \cdots < x_n = b \quad , \quad \Delta x = x_{i+1} - x_i = \frac{b-a}{n}$$

$$c = y_0 < y_1 < \cdots < y_n = d \quad , \quad \Delta y = y_{j+1} - y_j = \frac{d-c}{n}$$

$$p = z_0 < z_1 < \cdots < z_n = q \quad , \quad \Delta z = z_{k+1} - z_k = \frac{q-p}{n}$$

LET $f : B \to \mathbb{R}$ BE A BOUNDED FUNCTION ON B. A <u>RIEMANN SUM</u> FOR f IS AN EXPRESSION OF THE FORM

$$S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(c_{ijk}) \Delta V$$

WHERE $c_{ijk} \in B_{ijk} = [x_i, x_{i+1}] \times [y_j \times y_{j+1}] \times [z_k, z_{k+1}]$ ARE CHOSEN ARBITRARILY AND $\Delta V = \Delta x \Delta y \Delta z$

SUCH A FUNCTION $f : B \to \mathbb{R}$ IS SAID TO BE <u>INTEGRABLE OVER B</u> IF $S = \lim_{n\to\infty} S_n$ EXISTS AND IS INDEPENDENT OF THE CHOICES $C_{ijk} \in B_{ijk}$. IN THIS CASE WE WRITE

$$S = \iiint_B f(x,y,z)\,dV = \iiint_B f(x,y,z)\,\underbrace{dx\,dy\,dz}$$

OR 5 OTHER PERMUTATIONS OF THE SYMBOLS $\{dx, dy, dz\}$

<u>THEOREM</u>
IF THE DISCONTINUITIES OF f LIE IN THE UNION OF FINITELY MANY CONTINUOUS SURFACES IN B, THEN f IS INTEGRABLE ON B.

<u>Corollary</u>
IF f IS CONTINUOUS ON B THEN f IS INTEGRABLE ON B.

<u>THEOREM (FUBINI)</u>
UNDER THE ABOVE HYPOTHESIS, IF ANY OF THE 6 DISTINCT ITERATED INTEGRALS FOR f EXIST, THEN THEY ARE EQUAL TO

$$\iiint_B f\,dV$$

(AND TO EACH OTHER). IN PARTICULAR, IF f IS CONTINUOUS, ALL 6 ITERATED INTEGRALS ARE EQUAL.

FOR CONTINUOUS FUNCTIONS, WE CAN, AS BEFORE EXTEND THE DEFINITION OF INTEGRABILITY FROM RECTANGULAR REGIONS R TO 'ELEMENTARY' REGIONS W.

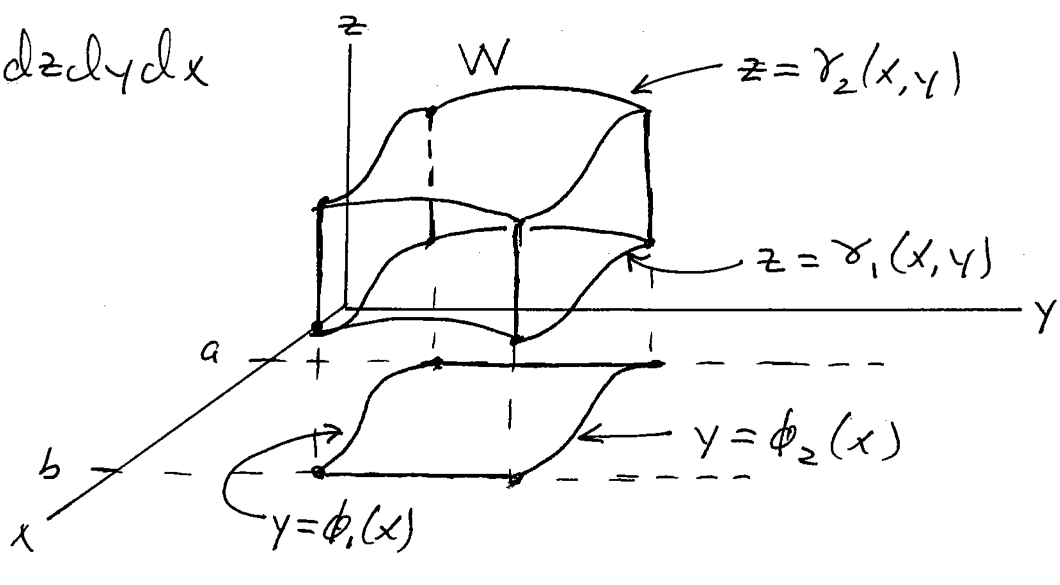
THERE ARE 6 TYPES OF ELEMENTARY REGIONS IN $\mathbb{R}^3$. WE ORGANIZE THESE INTO A HIERARCHY OF 3 MAIN TYPES, EACH WITH 2 SUB-TYPES. THE 6 TYPES ARE LABELED BY THE 6 OF $\{dx, dy, dz\}$.

(1) W IS BOUNDED BOTTOM & TOP BY GRAPHS $z=\gamma_1(x,y)$, $z=\gamma_2(x,y)$
 (1.1) PROJECTION OF W ONTO xy PLANE IS y-SIMPLE: $dz\,dy\,dx$
 (1.2) " " " " " " " x-SIMPLE: $dz\,dx\,dy$

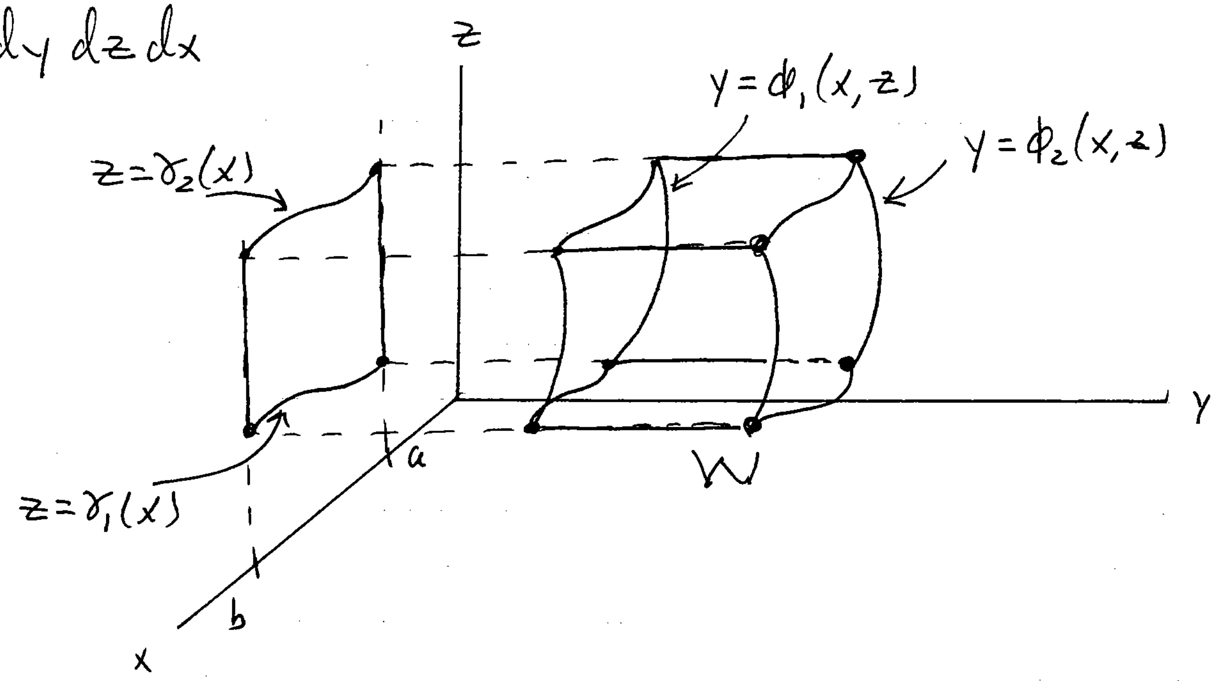
(2) W IS BOUNDED LEFT & RIGHT BY GRAPHS $y=\phi_1(x,z)$, $y=\phi_2(x,z)$
 (2.1) PROJECTION OF W ONTO xz PLANE IS z-SIMPLE: $dy\,dz\,dx$
 (2.2) " " " " " " " x-SIMPLE: $dy\,dx\,dz$

(3) W IS BOUNDED BACK & FRONT BY GRAPHS: $x=\phi_1(y,z)$, $x=\phi_2(y,z)$
 (3.1) PROJECTION OF W ONTO yz PLANE IS z-SIMPLE: $dx\,dz\,dy$
 (3.2) " " " " " " " y-SIMPLE: $dx\,dy\,dz$

Ex. $dz\,dy\,dx$



Ex. $dy\,dz\,dx$



EXERCISE
DRAW EXAMPLES (AS ABOVE) OF ALL 6 TYPES OF ELEMENTARY REGIONS.

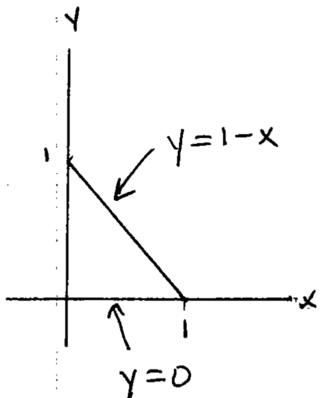
THEOREM
IF W IS AN ELEMENTARY REGION AND $f: W \to \mathbb{R}$ IS CONTINUOUS, THEN $\iiint_W f\,dV$ IS EQUAL TO AN APPROPRIATE ITERATED INTEGRAL. i.e

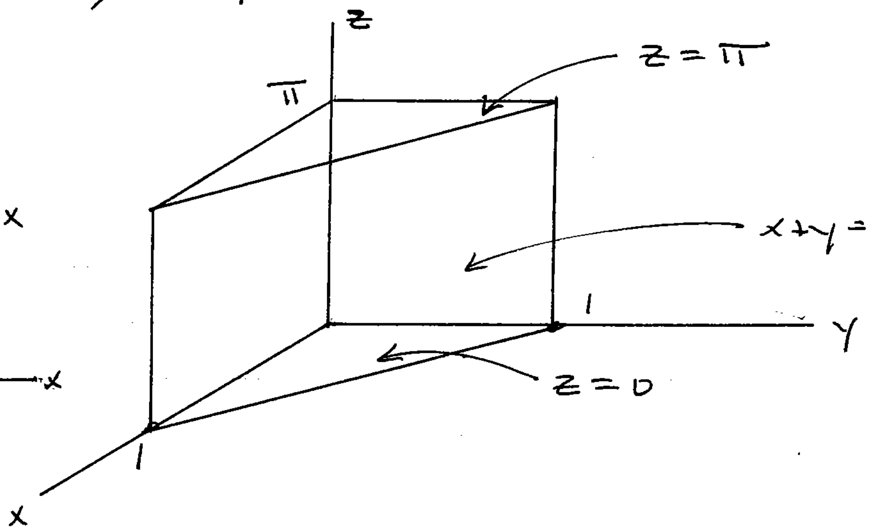
$$\iiint_W f\,dV = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

OR $$= \int_a^b \int_{\gamma_1(x)}^{\gamma_2(x)} \int_{\phi_1(x,z)}^{\phi_2(x,z)} f(x,y,z)\,dy\,dz\,dx$$

ETC....

EX. EVALUATE $\iiint_W x^2 \sin z \, dV$ WHERE W IS THE REGION BOUNDED BY: $z=0$, $z=\pi$, $y=0$, $y=1$, $x=0$, $x+y=1$





$$\int_0^1 \int_0^{1-x} \int_0^{\pi} x^2 \sin z \, dz\, dy\, dx = -\int_0^1 \int_0^{1-x} x^2 \cos z \Big|_0^{\pi} dy\, dx$$

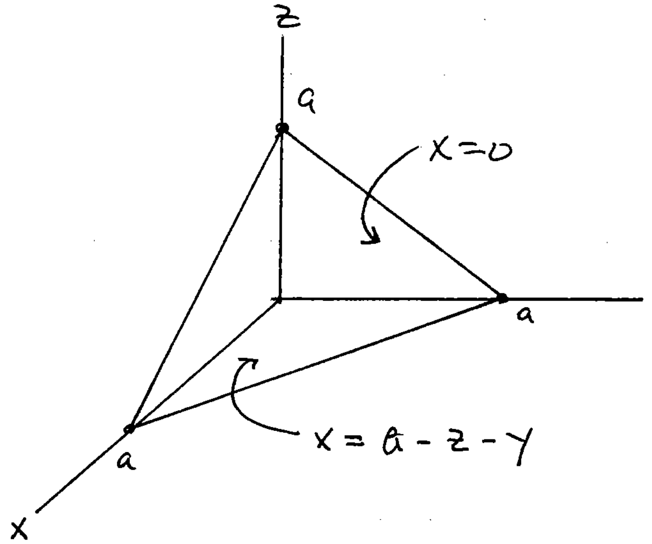
$$= -\int_0^1 \int_0^{1-x} x^2(-1-1)\, dy\, dx = 2 \int_0^1 x^2 y \Big|_0^{1-x} dx$$

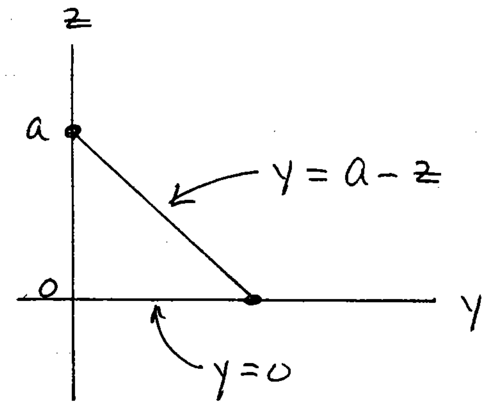
$$= 2 \int_0^1 (x^2 - x^3)\, dx = 2\left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_0^1$$

$$= 2\left(\frac{1}{3} - \frac{1}{4}\right) = 2 \cdot \frac{4-3}{12} = \boxed{\frac{1}{6}}$$

<u>EXERCISE</u> RE-DO THIS EXAMPLE 5 WAYS.

Ex. FIND $\iiint_W 10z^2\,dV$ WHERE W IS THE REGION BOUNDED BY $x=0$, $y=0$, $z=0$ AND $x+y+z=a$ (WHERE $a>0$).





BY CHOOSING $dx\,dy\,dz$, THE FIRST INTEGRATION IS TRIVIAL

$$\int_0^a\int_0^{a-z}\int_0^{a-z-y} 10z^2\,dx\,dy\,dz = \int_0^a\int_0^{a-z} 10z^2(a-z-y)\,dy\,dz$$

$$= \int_0^a 10z^2\left((a-z)y - \tfrac{1}{2}y^2\right)\Big|_0^{a-z}\,dz$$

$$= \int_0^a 10z^2\cdot\tfrac{1}{2}(a-z)^2\,dz = 5\int_0^a z^2(a^2-2az+z^2)\,dz$$

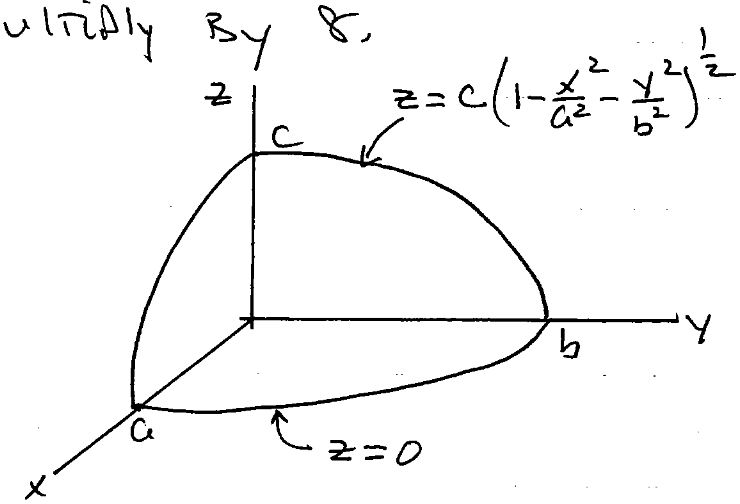
$$= 5\int_0^a (a^2z^2 - 2az^3 + z^4)\,dz = 5\left(\tfrac{1}{3}a^2z^3 - \tfrac{1}{2}az^4 + \tfrac{1}{5}z^5\right)\Big|_0^a$$

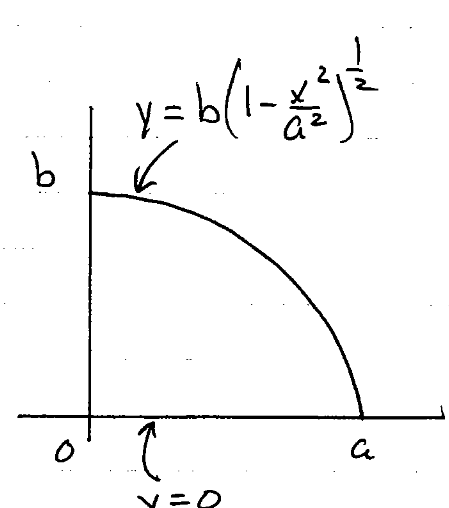
$$= 5\left(\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}\right)a^5 = 5\left(\tfrac{10-15+6}{30}\right)a^5 = \tfrac{5a^5}{30} = \boxed{\tfrac{a^5}{6}}$$

EX. DETERMINE THE VOLUME OF THE ELLIPSOID WITH SEMI-AXES a, b, c:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\quad}{c^2} = 1$$

THIS SOLID IS SYMMETRIC WITH RESPECT TO EACH OF THE COORDINATE PLANES. THUS WE MAY INTEGRATE $f(x,y,z) = 1$ OVER THAT PART OF THE ELLIPSOID IN THE FIRST OCTANT, THEN MULTIPLY BY 8.





TO SIMPLIFY NOTATION, LET

$$K = b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} = \frac{b}{a}\left(a^2 - x^2\right)^{\frac{1}{2}}$$

THEN

$$c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} = \frac{c}{b}\left(b^2\left(1 - \frac{x^2}{a^2}\right) - y^2\right)^{\frac{1}{2}} = \frac{c}{b}\left(K^2 - y^2\right)^{\frac{1}{2}}$$

OBSERVE K IS A FUNCTION OF x ONLY, HENCE IS CONSTANT WHEN INTEGRATING WRT. y & z.

THUS

$$\text{VOLUME} = 8\int_0^a \int_0^K \int_0^{\frac{c}{b}(K^2-y^2)^{\frac{1}{2}}} 1 \; dz\,dy\,dx$$

$$= 8\int_0^a \int_0^K \frac{c}{b}(K^2-y^2)^{\frac{1}{2}} \, dy\,dx$$

$$= \frac{8c}{b}\int_0^a \cancel{\frac{y}{2}(K^2-y^2)^{\frac{1}{2}}} + \frac{K^2}{2}\sin^{-1}\left(\frac{y}{K}\right)\Big|_0^K \, dx \quad \left\{\begin{matrix}\text{BY}\\ \text{FORMULA}\\ \#38\end{matrix}\right.$$

$$= \frac{8c}{b}\int_0^a \frac{K^2}{2}\cdot\frac{\pi}{2}\,dx = \frac{2\pi c}{b}\int_0^a \frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$= \frac{2\pi bc}{a^2}\int_0^a (a^2-x^2)\,dx = \frac{2\pi bc}{a^2}\left(a^2x - \frac{1}{3}x^3\right)\Big|_0^a$$

$$= \frac{2\pi bc}{a^2}\left(a^3 - \frac{1}{3}a^3\right) = 2\pi abc\cdot\frac{2}{3} = \boxed{\frac{4\pi}{3}abc}$$