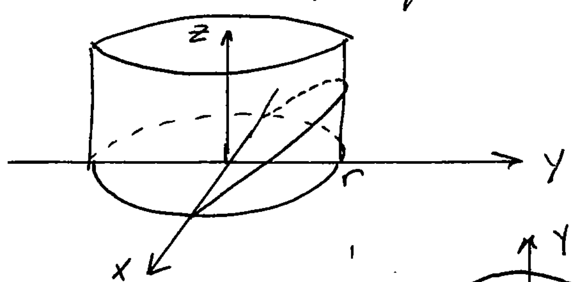
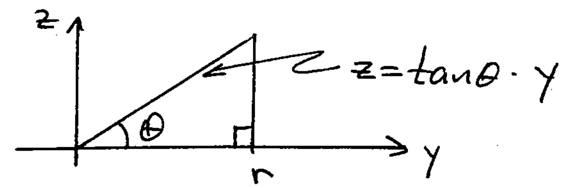
(5.4) CHANGING THE ORDER OF INTEGRATION

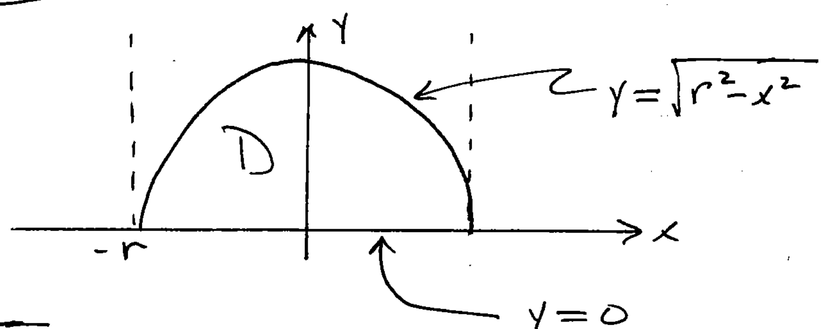
IF D IS BOTH y-SIMPLE AND x-SIMPLE THEN WE CAN INTEGRATE IN EITHER ORDER.

$$\iint_D f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$

EX. RECALL OUR VERY FIRST EXAMPLE: FIND THE VOLUME OF A CYLINDRICAL WEDGE.







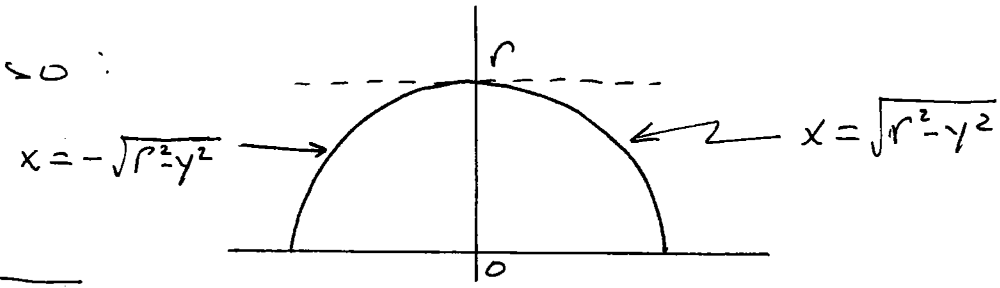
$$V = \int_{-r}^{r} \int_0^{\sqrt{r^2-x^2}} \tan\theta \cdot y\,dy\,dx = \int_{-r}^{r} \tfrac{1}{2}\tan\theta \cdot y^2 \Big|_0^{\sqrt{r^2-x^2}} dx$$

$$= \tfrac{1}{2}\tan\theta \int_{-r}^{r} (r^2 - x^2)\,dx = \tfrac{1}{2}\tan\theta \cdot \left(r^2 x - \tfrac{1}{3}x^3\right)\Big|_{-r}^{r}$$

$$= \tfrac{1}{2}\tan\theta \cdot \left(\left(r^3 - \tfrac{1}{3}r^3\right) - \left(-r^3 + \tfrac{1}{3}r^3\right)\right) = \tfrac{1}{\cancel{2}}\tan\theta \cdot \cancel{2} \cdot \tfrac{2}{3}r^3$$

$$= \boxed{\tfrac{2}{3} r^3 \tan\theta}$$

BUT ALSO:



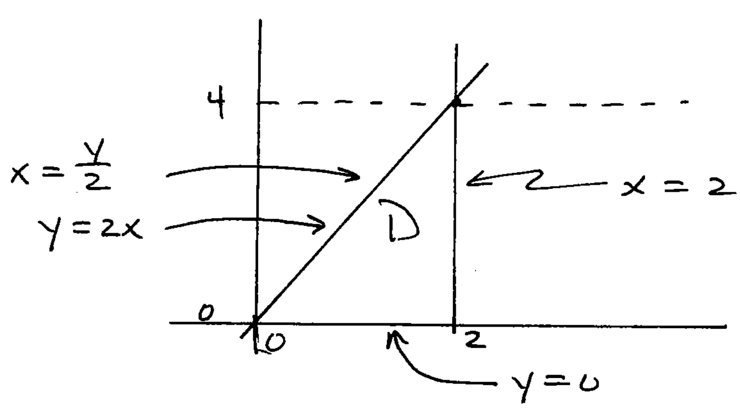
$$V=\int_0^r \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \tan\theta \cdot y \, dx \, dy = \int_0^r \tan\theta \cdot y x \Big|_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dy$$

$$= -\tan\theta \int_0^r (-2y)\sqrt{r^2-y^2} \, dy = -\tan\theta \cdot \tfrac{2}{3} \cdot (r^2-y^2)^{3/2} \Big|_0^r$$

$$= -\tan\theta \cdot \tfrac{2}{3} \left(0^{3/2} - (r^2)^{3/2} \right) = \boxed{\tfrac{2}{3} r^3 \tan\theta}.$$

IN THE LAST EXAMPLE THERE WAS NO PREFERED ORDER OF INTEGRATION. ON THE OTHER HAND, SOMETIMES THE INTEGRAL IS IMPOSSIBLE IN ONE ORDER, BUT EASY IN THE OTHER.

EX. EVALUATE $\displaystyle\int_0^4 \int_{y/2}^{2} e^{x^2} \, dx \, dy$



$$\begin{cases} 0 \le y \le 4 \\ \frac{y}{2} \le x \le 2 \end{cases}$$

OR

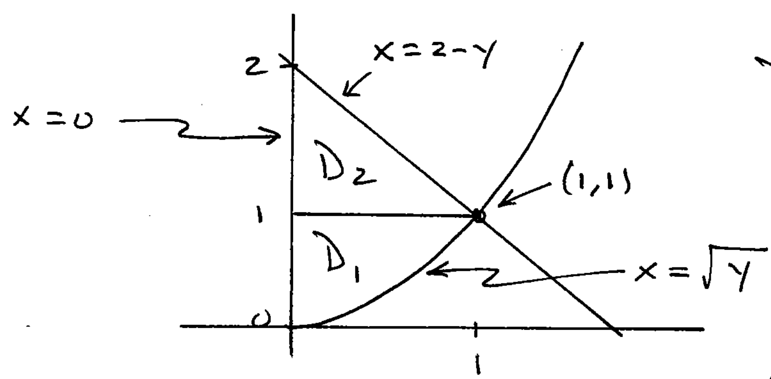
$$\begin{cases} 0 \le x \le 2 \\ 0 \le y \le 2x \end{cases}$$

$$\therefore \iint_D e^{x^2}\,dA = \int_0^2 \int_0^{2x} e^{x^2}\,dy\,dx = \int_0^2 y e^{x^2}\Big|_0^{2x}\,dx$$

$$= \int_0^2 2x e^{x^2}\,dx = e^{x^2}\Big|_0^2 = \boxed{e^4 - 1}$$

SOMETIMES A REGION WHICH IS NOT OF THE DESIRED TYPE CAN BE SPLIT INTO SEVERAL REGIONS OF THAT TYPE.

Ex. RECALL $D = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le 2-x\}$



$D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$

$$A(D) = A(D_1) + A(D_2) = \int_0^1 \int_0^{\sqrt{y}} dx\,dy + \underbrace{\int_1^2 \int_0^{2-y} dx\,dy}_{\frac{1}{2}}$$

$$= \int_0^1 y^{\frac{1}{2}}\,dy + \frac{1}{2} = \frac{2}{3} y^{3/2}\Big|_0^1 + \frac{1}{2}$$

$$= \frac{2}{3} + \frac{1}{2} = \frac{4+3}{6} = \boxed{\frac{7}{6}}$$

SUPPOSE $m \le f(x,y) \le M$ FOR ALL $(x,y) \in D$. THEN UPON INTEGRATING THE INEQUALITY OVER D WE GET

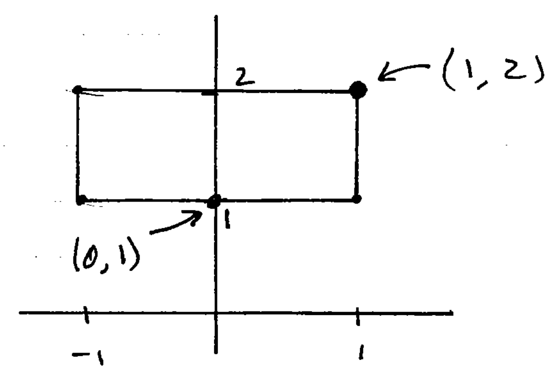
$$m \cdot A(D) \le \iint_D f(x,y)\,dA \le M \cdot A(D) . \quad (*)$$

THIS IS KNOWN AS THE <u>MEAN VALUE INEQUALITY</u>.

THIS INEQUALITY CAN BE USED TO ESTIMATE THE VALUE OF $\iint_D f\,dA$ WHEN IT CANNOT BE COMPUTED DIRECTLY.

<u>EX</u>. LET $D = [-1,1] \times [1,2]$

$$f(x,y) = \frac{2}{x^8 + y^3 + 1}$$



$\therefore\ m = \frac{2}{10} = \frac{1}{5},\ M = \frac{2}{2} = 1$

$$\therefore \quad \frac{2}{5} \le \iint_D \frac{2}{x^8 + y^3 + 1}\,dA \le 2$$

FOR SOME APPLICATIONS, THIS ESTIMATE MAY BE CLOSE ENOUGH.

MEAN VALUE THEOREM

LET D BE AN ELEMENTARY REGION AND SUPPOSE $f: D \to \mathbb{R}$ IS CONTINUOUS. THEN THERE EXISTS A POINT $(x_0, y_0) \in D$ SUCH THAT

$$\iint_D f(x,y)\,dA = f(x_0, y_0) \cdot A(D).$$

PROOF:

SINCE f IS CONTINUOUS AND D IS CLOSED, f ACHIEVES BOTH ITS MAXIMUM AND MINIMUM VALUES ON D, i.e. THERE EXIST POINTS (x_1, y_1) AND (x_2, y_2) IN D SUCH THAT

$$f(x_1, y_1) = m \quad \text{AND} \quad f(x_2, y_2) = M$$

WHERE M AND m ARE THE MAXIMUM AND MINIMUM VALUES OF f ON D, RESPECTIVELY.

DIVIDE INEQUALITY (*) THROUGH BY $A(D)$ TO GET

$$m \le \frac{1}{A(D)} \iint_D f(x,y)\,dA \le M.$$

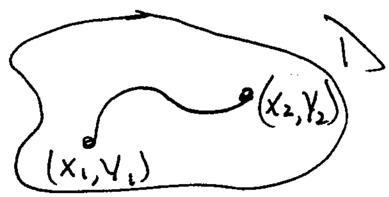
THE CONTINUOUS FUNCTION f MUST ACHIEVE EACH OF THE VALUES BETWEEN ITS MINIMUM m AND MAXIMUM M SOMEWHERE IN D.

THUS THERE EXISTS A POINT $(x_0, y_0) \in D$ SUCH THAT

$$f(x_0, y_0) = \frac{1}{A(D)} \iint_D f(x,y)\,dA \;,$$

WHICH IS THE DESIRED RESULT.

(THIS LAST STEP IS A CONSEQUENCE OF THE MVT FOR FUNCTIONS OF 1 VARIABLE AS WE NOW SHOW. LET $c : [0,1] \to D$ BE A CONTINUOUS CURVE IN D WITH $c(0) = (x_1, y_1)$ AND $c(1) = (x_2, y_2)$.



LET $g : [0,1] \to \mathbb{R}$ BE GIVEN BY $g(t) = f(c(t))$. THEN SINCE

$$g(0) \le \frac{1}{A(D)} \iint_D f\,dA \le g(1)$$

THERE EXISTS $t_0 \in [0,1]$ SUCH THAT $g(t_0) = \frac{1}{A(D)} \iint_D f\,dA$. LET $(x_0, y_0) = c(t_0)$.)

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NOTE:
THE NUMBER $\frac{1}{A(D)} \iint_D f\,dA$ IS THE AVERAGE VALUE OF f ON D.