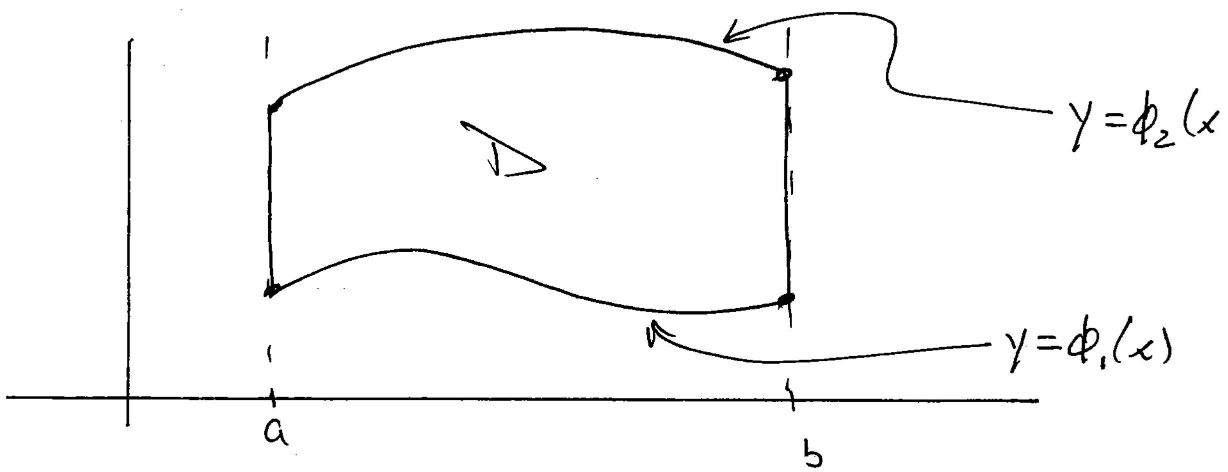
(5.3) INTEGRALS OVER MORE GENERAL REGIONS

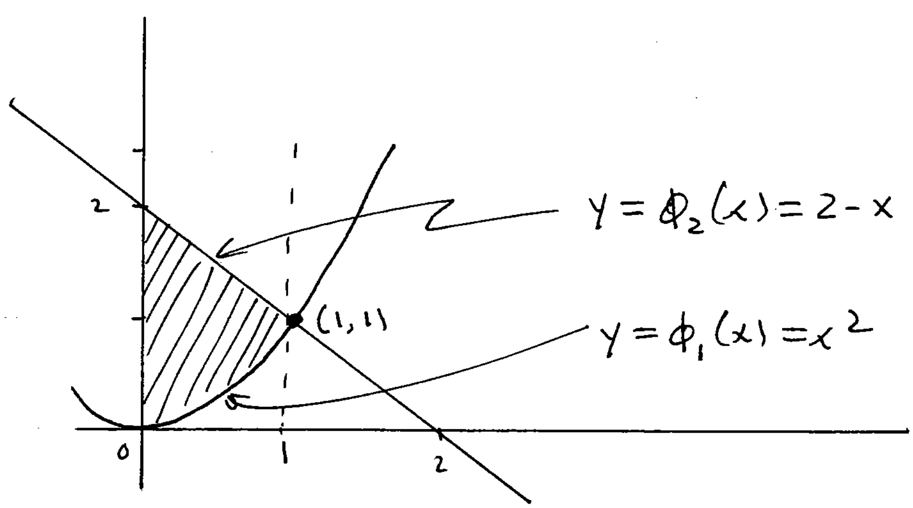
DEFN

A REGION $D \subseteq \mathbb{R}^2$ is called *y-simple* if THERE ARE CONTINUOUS FUNCTIONS $\phi_1 : [a,b] \to \mathbb{R}$ AND $\phi_2 : [a,b] \to \mathbb{R}$ SUCH THAT

$$D = \{(x,y) \mid a \le x \le b \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$$

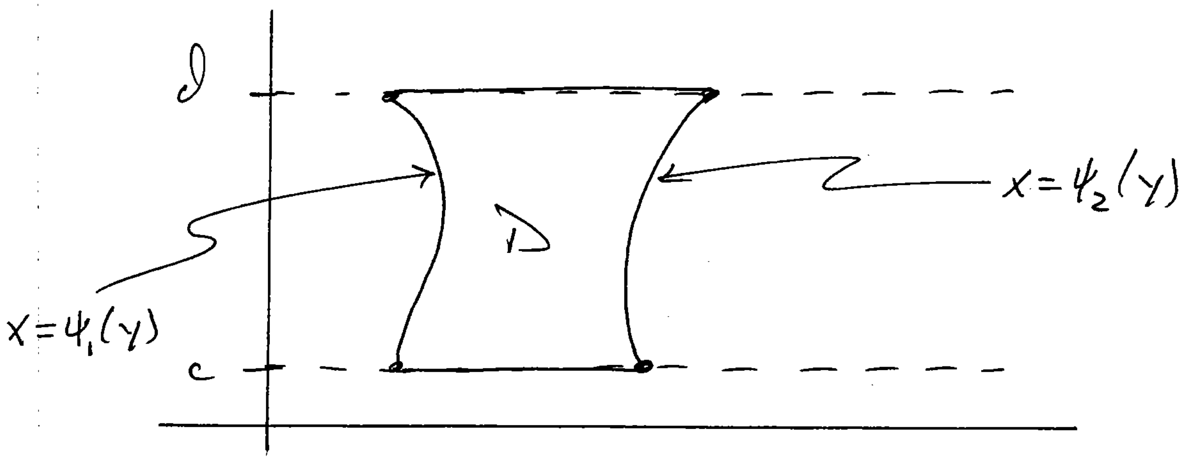


Ex. $D = \{(x,y) \mid 0 \le x \le 1 \ \& \ x^2 \le y \le 2-x\}$

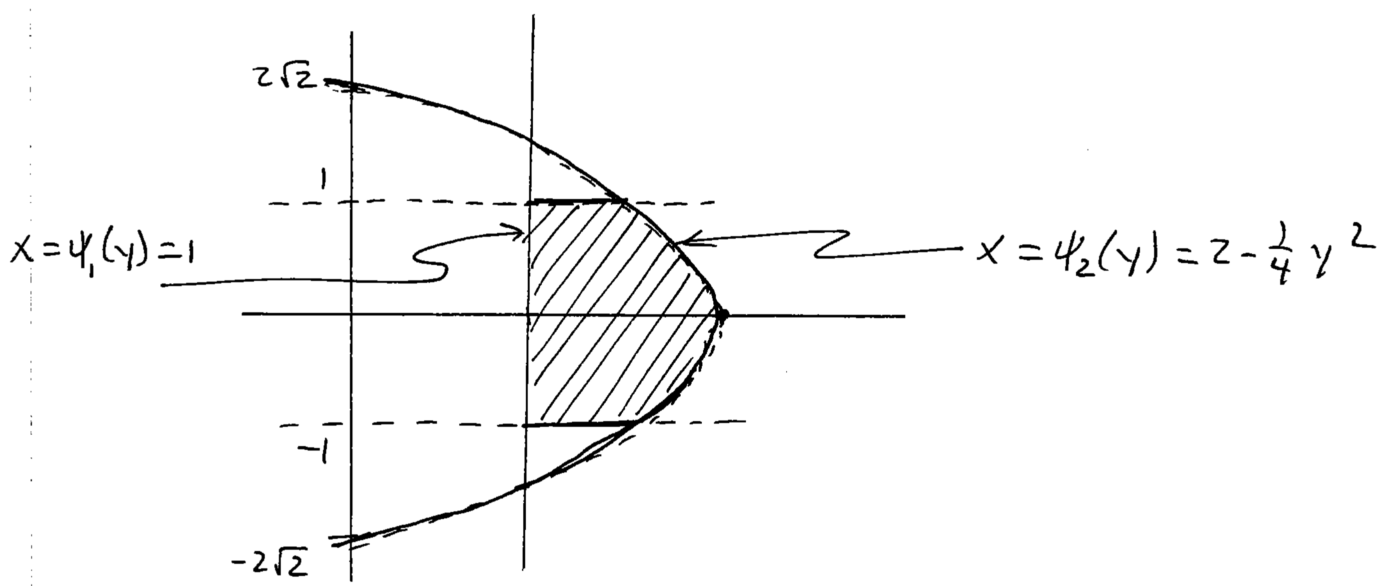


<u>DEFN.</u>
$D \subseteq \mathbb{R}^2$ IS CALLED <u>X-SIMPLE</u> IF THERE ARE CONTINUOUS FUNCTIONS $\psi_1 : [c,d] \to \mathbb{R}$ AND $\psi_2 : [c,d] \to \mathbb{R}$ SUCH THAT

$$D = \{ (x,y) \mid c \le d \le y \text{ \& } \psi_1(y) \le x \le \psi_2(y) \}$$

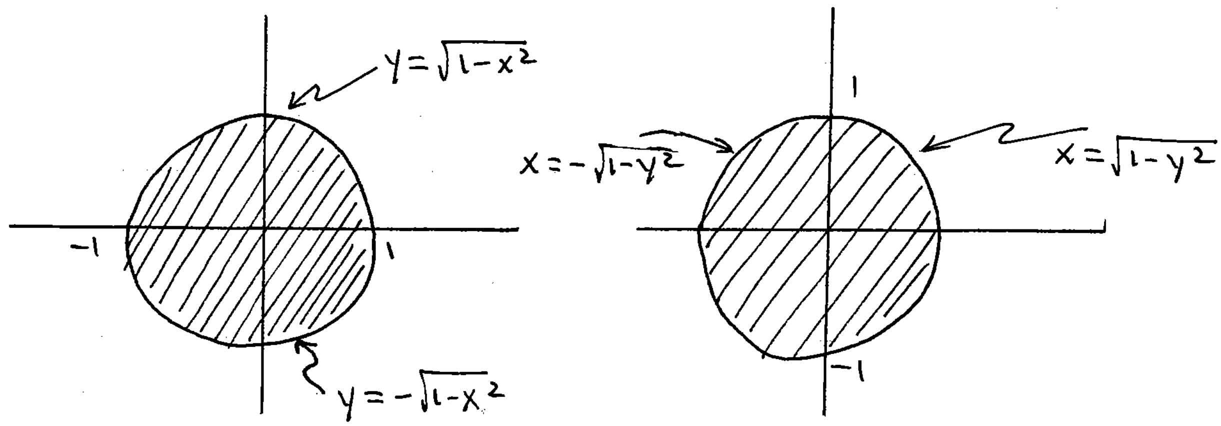


<u>EX.</u> $D = \{ (x,y) \mid -1 \le y \le 1 \text{ \& } 1 \le x \le 2 - \frac{1}{4}y^2 \}$



NOTE THAT THE FIRST EXAMPLE WAS Y-SIMPLE BUT NOT X-SIMPLE, WHILE THE SECOND WAS X-SIMPLE BUT NOT Y-SIMPLE. THE FOLLOWING EXAMPLE IS BOTH.

<u>EX.</u> D IS THE INTERIOR OF THE UNIT CIRCLE CENTERED AT (0,0).



A REGION WHICH IS EITHER Y-SIMPLE OR X-SIMPLE (OR BOTH) WILL BE CALLED AN <u>ELEMENTARY REGION</u>.

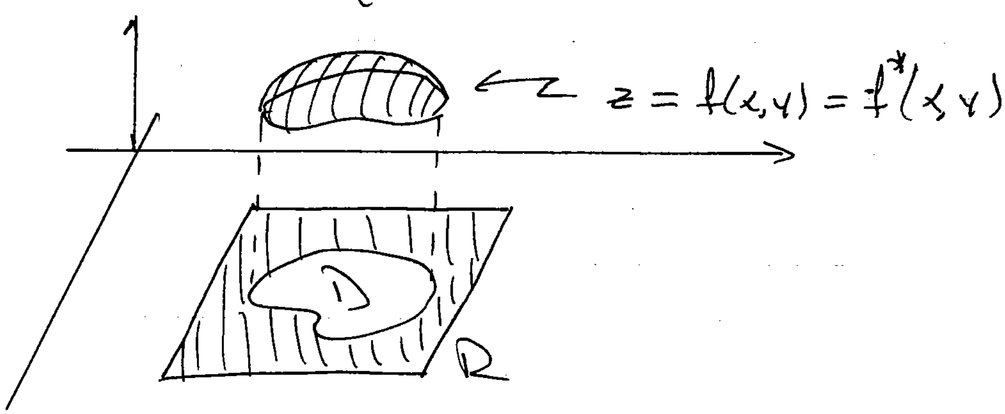
NOTE THAT ANY ELEMENTARY REGION HAS A BOUNDARY (DENOTED ∂D) WHICH IS A FINITE UNION OF CONTINUOUS CURVES IN $\mathbb{R}^2$.

WE WISH TO EXTEND OUR DEFINITION OF INTEGRAL TO FUNCTIONS DEFINED ON ELEMENTARY REGIONS.

DEFN.
LET $D \subseteq \mathbb{R}^2$ BE ELEMENTARY, AND LET $R \subseteq \mathbb{R}^2$ BE A RECTANGLE (WHOSE SIDES ARE PARALLEL TO THE COORDINATE AXES) WHICH CONTAINS D.

LET $f: D \to \mathbb{R}$ BE CONTINUOUS (HENCE BOUNDED) AND DEFINE $f^*: R \to \mathbb{R}$ BY

$$f^*(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R - D. \end{cases}$$



OBSERVE THAT f^* IS BOUNDED ON R (SINCE f IS BOUNDED ON D), AND THAT THE POINTS OF DISCONTINUITY OF f^* LIE ON ∂D.
$\therefore f^*$ IS INTEGRABLE ON R.

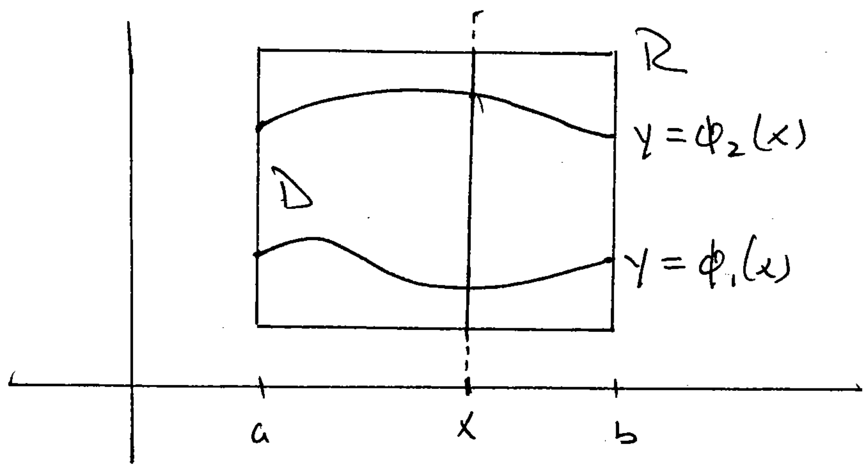
NOW DEFINE

$$\iint_D f(x,y)\,dA = \iint_R f^*(x,y)\,dA.$$

THEOREM

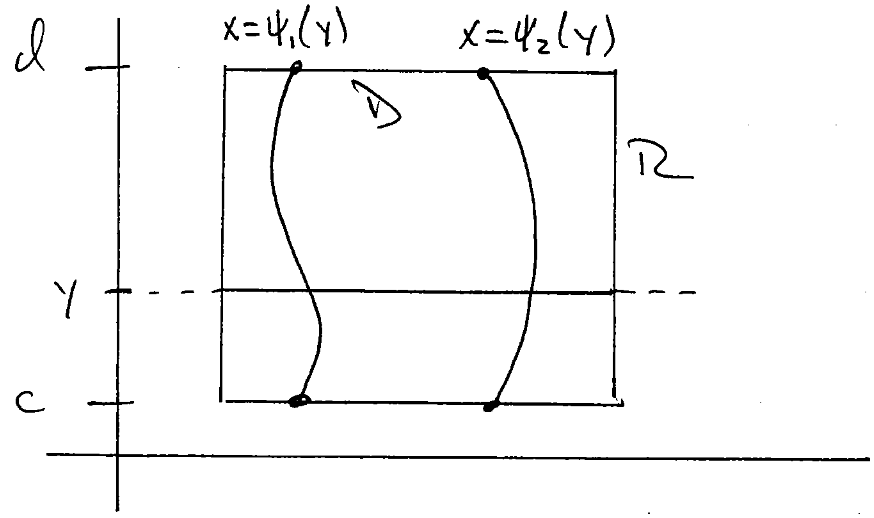
IF D is y-SIMPLE, THEN

$$\iint_D f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

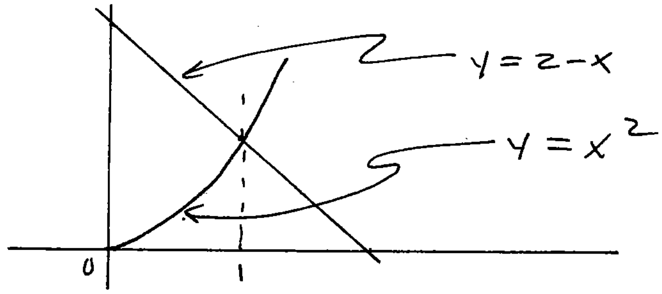


SIMILARLY IF D is x-SIMPLE, THEN

$$\iint_D f(x,y)\,dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$



Ex. $f(x,y)=xy$, $0 \le x \le 1$, $x^2 \le y \le 2-x$



$$\iint_D xy\,dA = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1 \frac{1}{2}xy^2 \Big|_{x^2}^{2-x} dx$$

$$= \int_0^1 \frac{1}{2}x\left((2-x)^2 - (x^2)^2\right) dx$$

$$= \frac{1}{2}\int_0^1 x(4-4x+x^2-x^4)\,dx$$

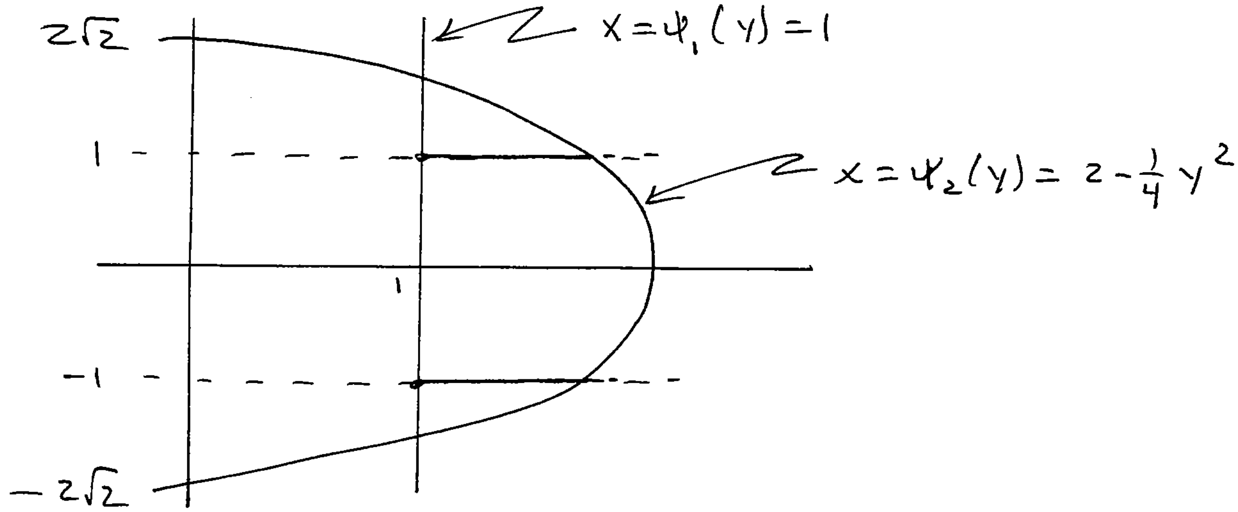
$$= \frac{1}{2}\int_0^1 (4x-4x^2+x^3-x^5)\,dx$$

$$= \frac{1}{2}\left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 - \frac{1}{6}x^6\right)\Big|_0^1$$

$$= \frac{1}{2}\left(2-\frac{4}{3}+\frac{1}{4}-\frac{1}{6}\right) = \frac{1}{2}\left(\frac{24-16+3-2}{12}\right) = \frac{1}{2}\cdot\frac{9}{12}$$

$$= \frac{1}{2}\cdot\frac{3}{4} = \boxed{\frac{3}{8}}.$$

Ex. $f(x,y) = y^2$, $-1 \le y \le 1$, $1 \le x \le 2 - \frac{1}{4}y^2$



$$\iint_D y^2\,dA = \int_{-1}^{1}\int_{1}^{2-\frac{1}{4}y^2} y^2\,dx\,dy = \int_{-1}^{1} y^2x\Big|_1^{2-\frac{1}{4}y^2}\,dy$$

$$= \int_{-1}^{1} y^2\left(\left(2-\tfrac{1}{4}y^2\right)-1\right)dy = \int_{-1}^{1} y^2\left(1-\tfrac{1}{4}y^2\right)dy$$

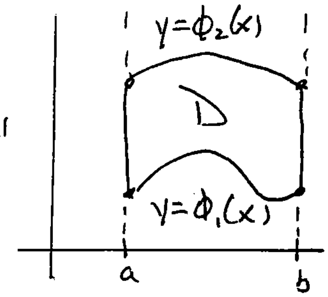
$$= \int_{-1}^{1}\left(y^2-\tfrac{1}{4}y^4\right)dy = \tfrac{1}{3}y^3-\tfrac{1}{20}y^5\Big|_{-1}^{1}$$

$$= \left(\tfrac{1}{3}-\tfrac{1}{20}\right)-\left(-\tfrac{1}{3}+\tfrac{1}{20}\right) = \tfrac{2}{3}-\tfrac{2}{20}$$

$$= \tfrac{2}{3}-\tfrac{1}{10} = \tfrac{20-3}{30} = \boxed{\tfrac{17}{30}}$$

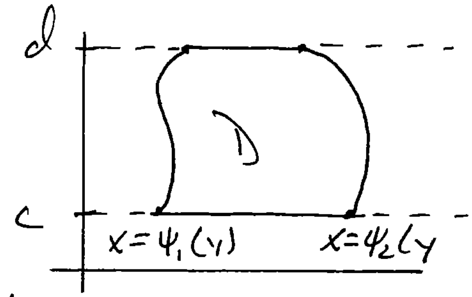
NOTICE THAT THE AREA OF AN ELEMENTARY REGION D CAN BE OBTAINED BY INTEGRATING THE FUNCTION $f(x,y)=1$ OVER D, SINCE IN THIS CASE $A(D)$ = VOLUME OVER D UNDER THE SURFACE $z=1$.

THUS IF D IS y-SIMPLE THEN



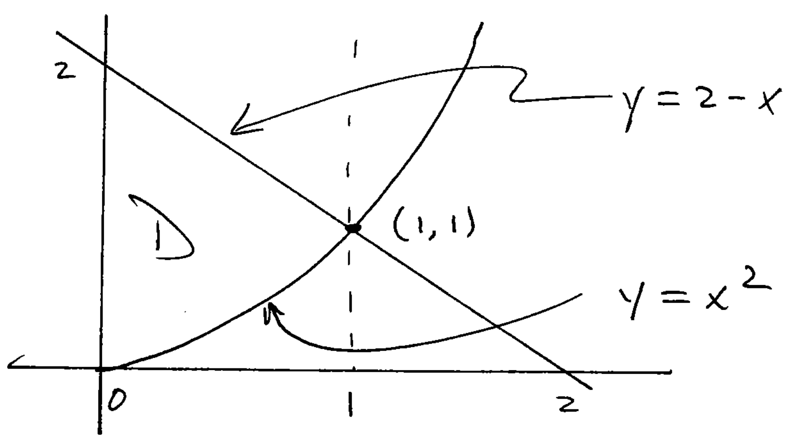
$$A(D)=\iint_D dA=\int_a^b\int_{\phi_1(x)}^{\phi_2(x)} dy\,dx=\int_a^b(\phi_2(x)-\phi_1(x))\,dx$$

AND IF D IS x-SIMPLE THEN



$$A(D)=\iint_D dA=\int_c^d\int_{\psi_1(y)}^{\psi_2(y)} dx\,dy=\int_c^d(\psi_2(y)-\psi_1(y))\,dy$$

EX. CONSIDER AGAIN THE REGION D GIVEN BY $0\le x\le 1$ AND $x^2\le y\le 2-x$, WHICH IS y-SIMPLE



THUS

$$A(D) = \int_0^1 \int_{x^2}^{2-x} dy\,dx = \int_0^1 y \Big|_{x^2}^{2-x} = \int_0^1 (2-x-x^2)\,dx$$

$$= \left(2x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_0^1 = 2 - \tfrac{1}{2} - \tfrac{1}{3} =$$

$$= \frac{12-3-2}{6} = \boxed{\frac{7}{6}}$$