(5.2) DEFINITION OF THE INTEGRAL

THE PRECEDING DISCUSSION WAS SOMEWHAT LESS THAN RIGOROUS SINCE WE NEVER GAVE A PRECISE DEFINITION OF VOLUME

HERE WE GIVE A RIGOROUS DEFINITION OF INTEGRAL, FROM WHICH THE DEFINITION OF VOLUME FOLLOWS.

LET $R = [a,b] \times [c,d]$ BE A CLOSED RECTANGLE. A <u>REGULAR PARTION</u> OF R OF ORDER n CONSISTS OF TWO COLLECTIONS $\{x_i\}$, $\{y_j\}$ OF $n+1$ EQUALLY SPACED POINTS:

$$a = x_0 < x_1 < \cdots < x_n = b \qquad \Delta x = x_{i+1} - x_i = \frac{b-a}{n}$$

$$c = y_0 < y_1 < \cdots < y_n = d \qquad \Delta y = y_{j+1} - y_j = \frac{d-c}{n}$$

LET $f : R \to \mathbb{R}$ BE A <u>BOUNDED</u> FUNCTION, i.e. THERE EXISTS $M > 0$ SUCH THAT

$$|f(x,y)| < M \quad \text{for all } (x,y) \in R$$

NOTE THAT IF f IS CONTINUOUS, THEN f IS BOUNDED SINCE R IS CLOSED. THIS FALSE IF WE REPLACE R BY A NON-CLOSED SET.

LET R_{ij} BE THE CLOSED RECTANGLE

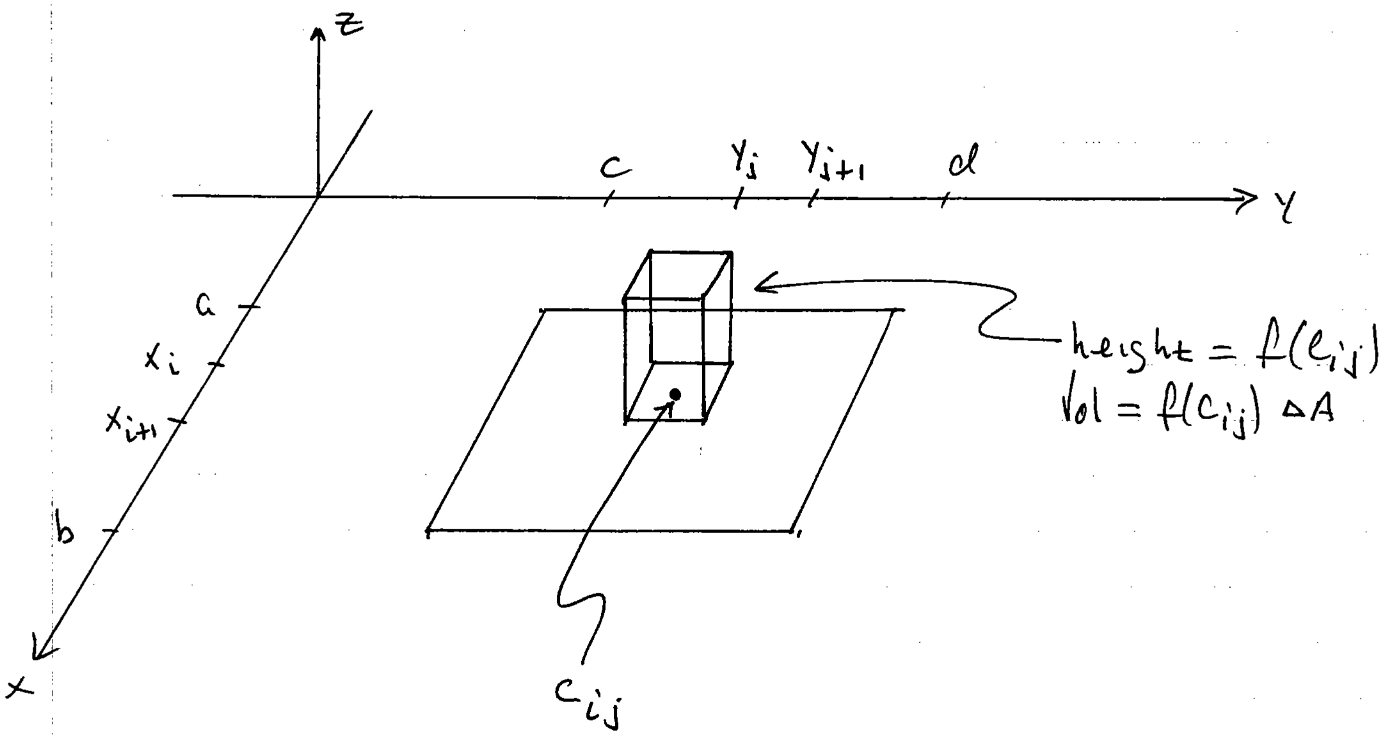
$$R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \quad \begin{cases} 0 \le i \le n-1 \\ 0 \le j \le n-1 \end{cases}$$

WHOSE AREA IS $\Delta A = \Delta x \Delta y$. CHOOSE POINTS $c_{ij} \in R_{ij}$ ARBITRARILY.

A <u>RIEMANN SUM</u> FOR f OVER R IS GIVEN BY

$$S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(c_{ij}) \Delta A = \sum_{i,j=0}^{n-1} f(c_{ij}) \Delta x \Delta y$$

OBSERVE THAT S_n DEPENDS NOT ONLY ON n, BUT ALSO ON THE PARTICULAR POINTS c_{ij} CHOSEN FROM R_{ij}



DEFN.

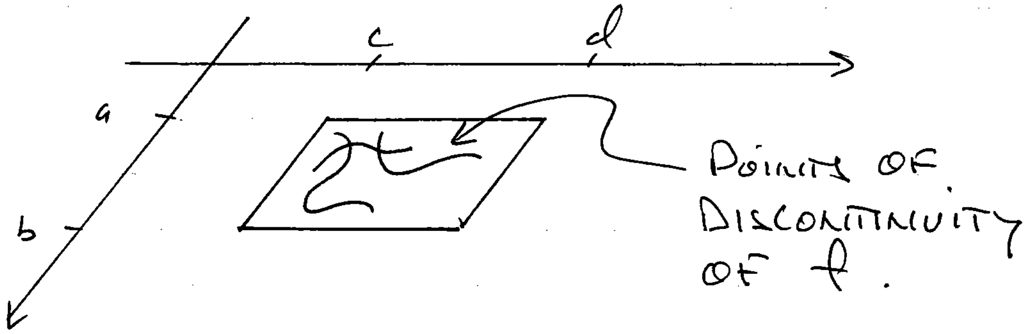
IF $S = \lim_{n\to\infty} S_n$ EXISTS AND IS THE SAME FOR ANY CHOICE OF POINTS $c_{ij} \in R_{ij}$, THEN WE SAY THAT f IS INTEGRABLE OVER R AND WE WRITE

$$S = \iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx .$$

NOTICE THAT WE DID NOT ASSUME THAT $f(x,y) \geq 0$ FOR $(x,y) \in R$. IF IT SO HAPPENS THAT f IS NON-NEGATIVE ON R, THEN S HAS AN OBVIOUS INTERPRETATION AS VOLUME. THUS VOLUME IS DEFINED VIA THE INTEGRAL INSTEAD OF THE OTHER WAY AROUND.

THEOREM

IF $f : R \to \mathbb{R}$ IS BOUNDED, AND THE DISCONTINUITIES OF f LIE IN THE UNION OF FINITELY MANY CONTINUOUS CURVES IN R, THEN f IS INTEGRABLE OVER R.



Corollary

IF $f: R \to \mathbb{R}$ is CONTINUOUS, THEN f is INTEGRABLE OVER R.

Theorem

LET f AND g BE INTEGRABLE FUNCTIONS ON R, AND LET c BE A CONSTANT. THEN $f+g$ AND $c \cdot f$ ARE INTEGRABLE FUNCTIONS, AND

(1) $$\iint_R (f+g)\, dA = \iint_R f\, dA + \iint_R g\, dA$$

(2) $$\iint_R c f\, dA = c \cdot \iint_R f\, dA$$

(3) if $f(x,y) \geq g(x,y)$ FOR All $(x,y) \in R$, THEN
$$\iint_R f\, dA \geq \iint_R g\, dA$$

(4) if R_i $(1 \leq i \leq m)$ ARE PAIRWISE DISJOINT RECTANGLES ON WHICH f is INTEGRABLE, AND if $Q = R_1 \cup \cdots \cup R_m$ is A RECTANGLE, THEN f is INTEGRABLE OVER Q AND

$$\iint_Q f\, dA = \sum_{i=1}^{m} \iint_{R_i} f\, dA$$

Fubini's Theorem

If $f: R \to \mathbb{R}$ is bounded, and the discontinuities of f lie in the union of finitely many continuous curves in R, then:

(1) If $\int_c^d f(x,y)\,dy$ exists for all $x \in [a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx .$$

(2) If $\int_a^b f(x,y)\,dx$ exists for all $y \in [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy .$$

In particular, if both (1) and (2) hold simultaneously, then the two iterated integrals for f over R are equal.

Corollary

If $f: R \to \mathbb{R}$ is continuous then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx .$$

Fubini's Theorem and its Corollary say that under certain mild conditions, one can swap the order of integration in a double integral, i.e. $dx\,dy$ vs. $dy\,dx$.

ALL OF THESE THEOREMS ARE PROVED BY APPEALING TO THE DEFINITION OF THE INTEGRAL AS A LIMIT OF RIEMANN SUMS.

FOR INSTANCE, THE EQUALITY OF ITERATED FINITE SUMS

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_{ij} = \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} a_{ij}$$

IS THE KEY STEP IN THE PROOF OF FUBINI'S THEOREM. (JUST APPLY THE ABOVE FACT TO THE RIEMANN SUMS FOR f OVER R.)

Ex. $\iint_R y e^{xy}\, dA$, $R = [0,1] \times [0,1]$.

$$\int_0^1 \int_0^1 y e^{xy}\, dx\, dy = \int_0^1 e^{xy}\Big|_0^1 \, dy$$

$$= \int_0^1 (e^y - 1)\, dy$$

$$= (e^y - y)\Big|_0^1 = (e-1) - (1-0)$$

$$= e.$$

$\int_0^1 \int_0^1 y e^{xy}\, dy\, dx$ is messy!

EX. $\displaystyle\iint_R 2x\sin(xy)\cos(xy)\,dA$, $R=[0,1]\times[0,\frac{\pi}{2}]$

$$\int_0^1\int_0^{\pi/2} 2x\sin(xy)\cos(xy)\,dy\,dx = \int_0^1 \sin^2(xy)\Big|_0^{\pi/2}\,dx$$

$$= \int_0^1 \sin^2\left(\frac{\pi}{2}x\right)dx$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}x \quad \sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}x\right)\right)\Big|_0^1 \qquad (\#14)$$

$$= \frac{1}{\pi}\left(\left(\frac{\pi}{2}-0\right)-(0-0)\right) = \frac{1}{2}.$$

$\displaystyle\int_0^{\frac{\pi}{2}}\int_0^1 2x\sin(xy)\cos(xy)\,dx\,dy$ is messy!

EX. $R=[0,1]\times[0,1]$, $f(x,y)=\begin{cases}1 & x \text{ RATIONAL}\\ 2y & x \text{ IRRATIONAL}\end{cases}$

$$\int_0^1\int_0^1 f(x,y)\,dy\,dx = \int_0^1 \begin{Bmatrix} y \\ y^2\end{Bmatrix}_0^1 dx = \int_0^1 \begin{Bmatrix}1-0\\1^2-0^2\end{Bmatrix} dx$$

$$= \int_0^1 dx = x\Big|_0^1 = 1$$

BUT ONE CHECKS THAT $\int_0^1 f(x,y)\,dx$ DOES NOT EXIST FOR ANY $y \in [0,1]$.

$$\therefore \int_0^1\int_0^1 f(x,y)\,dx\,dy \quad \text{D.N.E.}$$

ONE ALSO CHECKS DIRECTLY THAT

$$\iint_R f(x,y)\,dA \quad \text{D.N.E.}$$

INDEED, BY CHOOSING THE POINTS c_{ij} IN THE RIEMANN SUMS FOR f TO HAVE ALTERNATELY RATIONAL OR IRRATIONAL X-COORDINATES, WE CAN CHANGE THE VALUES OF THE SUMS, AND ALSO THEIR LIMIT. HENCE f IS NOT INTEGRABLE ON R.