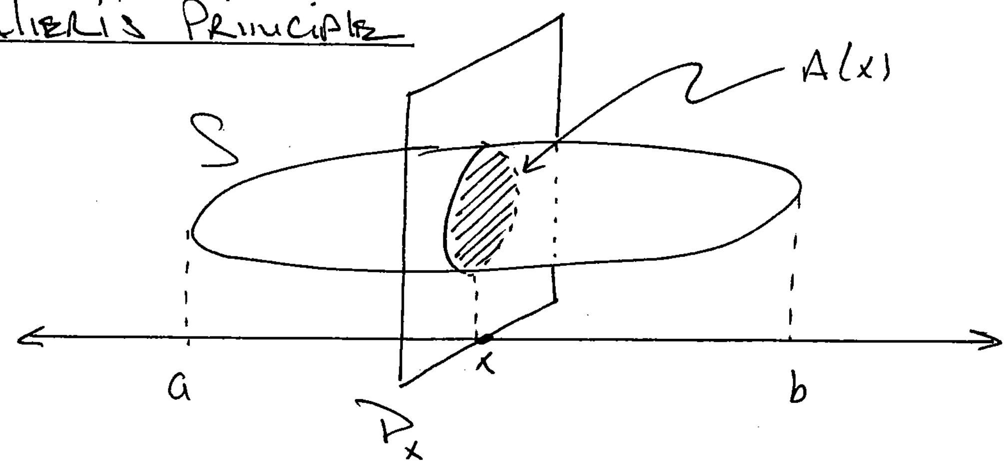
MATH 23B
MULTIVARIABLE CALCULUS
SPRING 2006

(5.1) INTRODUCTION

WE BEGIN WITH THE FOLLOWING GEOMETRICAL PROBLEM: DETERMINE THE VOLUME OF A SOLID REGION IN $\mathbb{R}^3$.

CAVALIERI'S PRINCIPLE



LET S BE A SOLID REGION, AND LET P_x $(a \le x \le b)$ BE A FAMILY OF PLANES PERPENDICULAR TO THE x-AXIS, AND SUCH THAT:

(1) S LIES BETWEEN P_a AND P_b

(2) THE AREA OF $S \cap P_x$ IS $A(x)$.

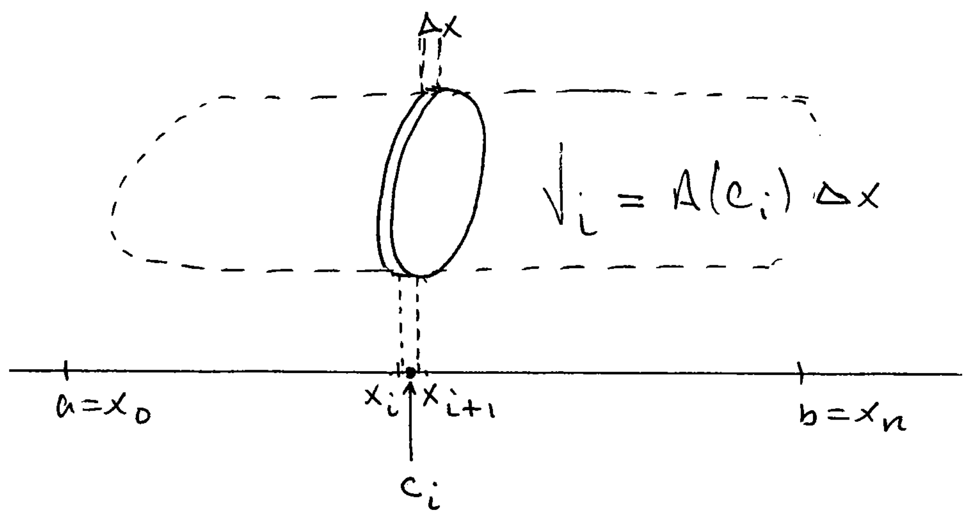
THEN

$$\text{vol}(S) = \int_a^b A(x)\, dx .$$

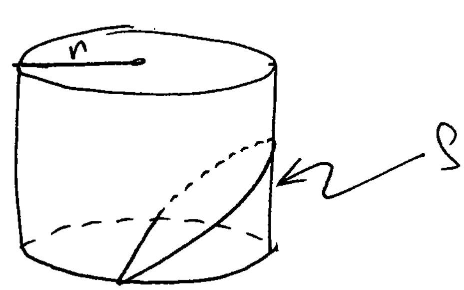
TO JUSTIFY THIS PRINCIPLE, RECALL THAT THE INTEGRAL

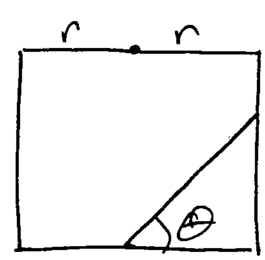
$\int_a^b A(x)\,dx$ is defined as the limit of Riemann sums: $\lim_{n\to\infty} \sum_{i=0}^{n-1} A(c_i)(x_{i+1}-x_i)$

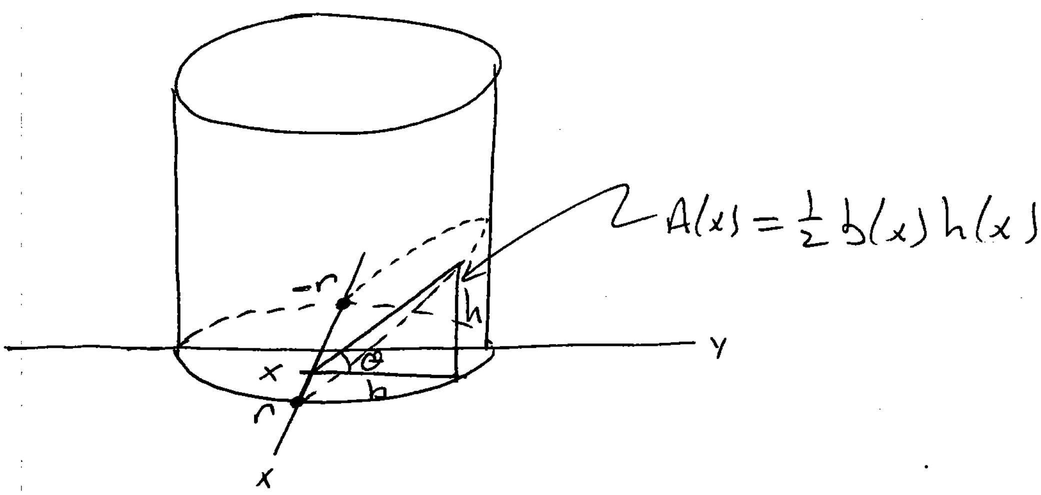
where $a = x_0 < x_1 < \dots < x_n = b$ are $n+1$ equally spaced points and c_i is any point in $[x_i, x_{i+1}]$. Each term is the volume of a small 'cylinder' of width $\Delta x = x_{i+1} - x_i = \frac{b-a}{n}$ and base area $A(c_i)$

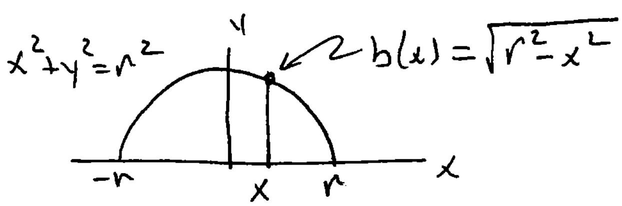


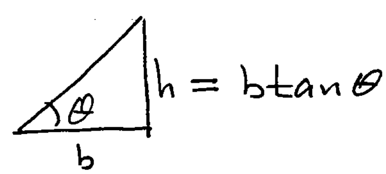
Example: Determine the volume of a wedge of a circular cylinder obtained by passing a plane through a diameter of the base circle at an angle θ from the horizontal.











$$\therefore \quad A(x) = \frac{1}{2}\, b(x)\cdot b(x)\tan\theta = \frac{\tan\theta}{2}(r^2-x^2)$$

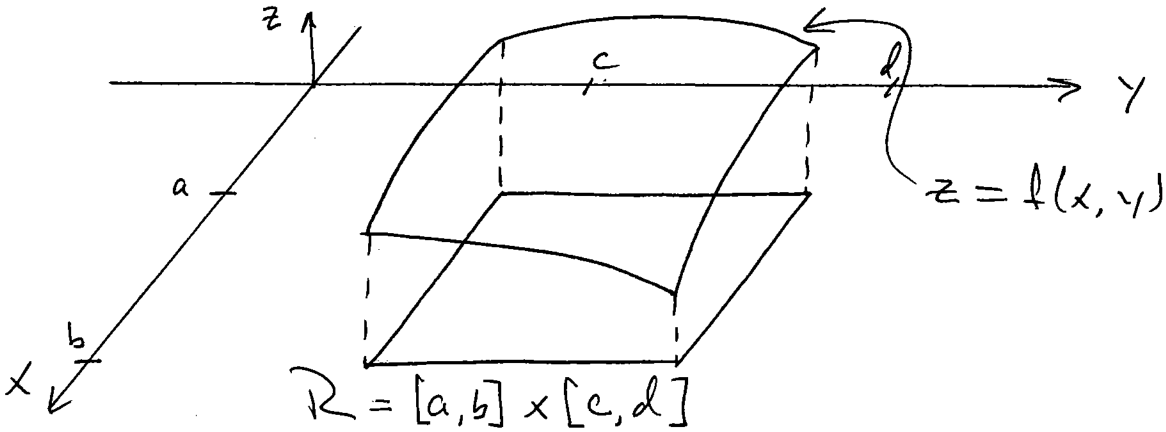
$$\therefore \quad V = \int_{-r}^{r} A(x)\,dx = \frac{\tan\theta}{2}\cdot 2\int_0^r (r^2-x^2)\,dx$$

$$= \tan\theta\left(r^2x-\frac{x^3}{3}\right)\Big|_0^r = \tan\theta\left(r^3-\frac{1}{3}r^3\right)$$

$$= \frac{2}{3}r^3\tan\theta \,. \qquad \blacksquare$$

WE NOW EXAMINE A SPECIAL CASE OF VOLUME OF A SOLID REGION.

LET $f : R \subseteq \mathbb{R}^2 \to \mathbb{R}$ BE A CONTINUOUS FUNCTION DEFINED ON A RECTANGLE R IN THE xy PLANE WHOSE SIDES ARE PARALLEL TO THE COORDINATE AXES.



$R = [a,b] \times [c,d]$

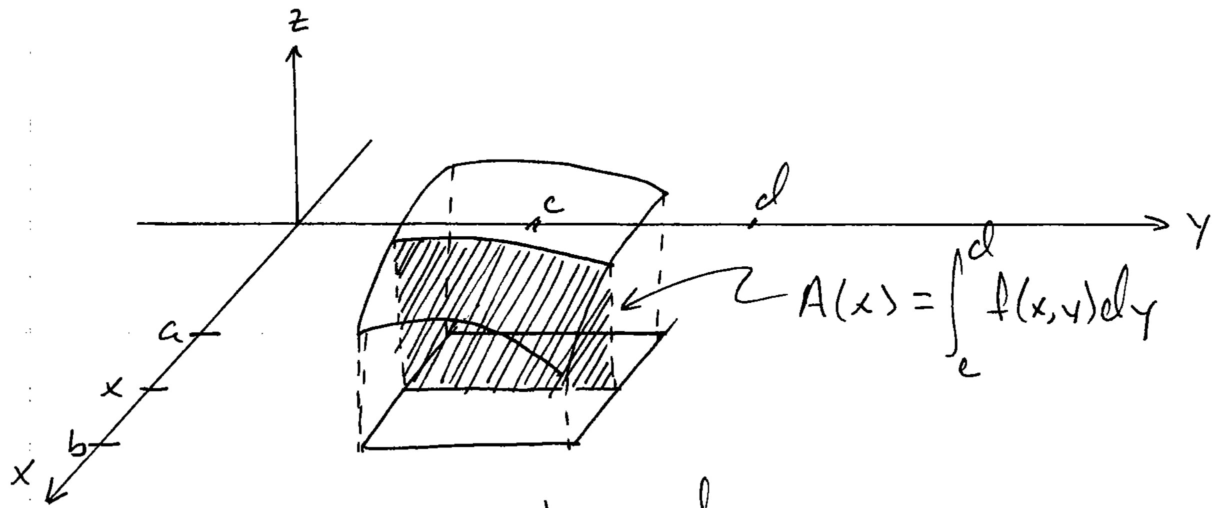
ASSUME $f(x,y) \geq 0$ FOR $(x,y) \in R$ SO THE GRAPH $z = f(x,y)$ LIES ABOVE THE xy PLANE.

<u>DEFN</u>

THE VOLUME OF THE SOLID REGION LYING UNDER THE GRAPH $z = f(x,y)$ AND ABOVE R IS CALLED THE <u>DOUBLE INTEGRAL OF f OVER R</u> AND IS DENOTED

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \iint_R f(x,y)\,dy\,dx$$

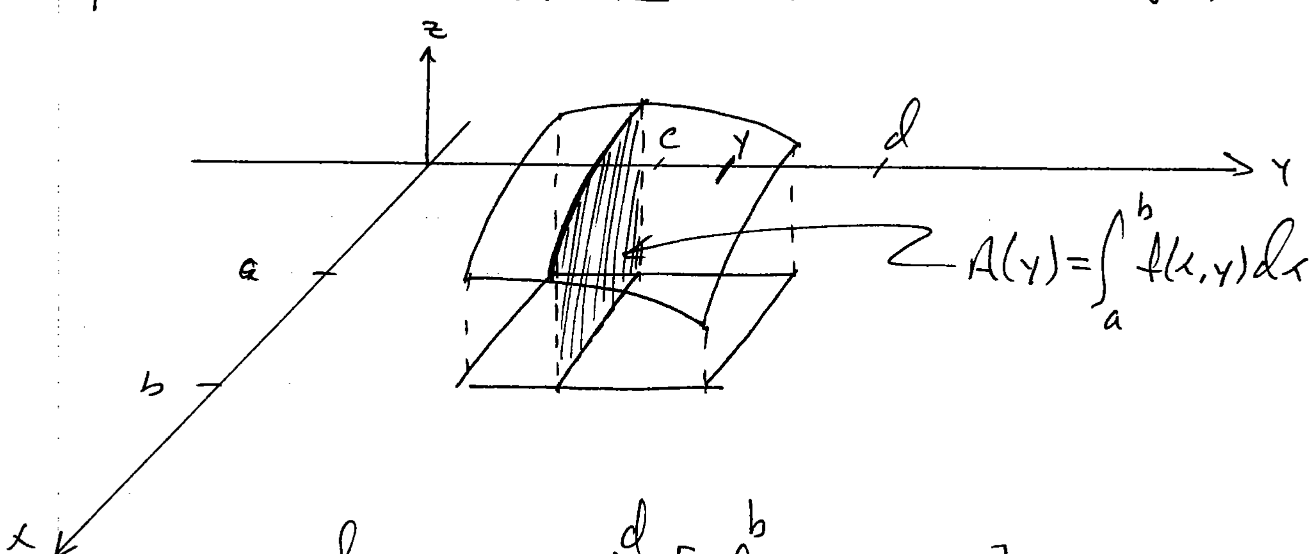
OBSERVE THAT CAVALIERI PRINCIPLE CAN BE USED TO EVALUATE SUCH A VOLUME.



$$A(x) = \int_c^d f(x,y)\,dy$$

$$\therefore\ V = \int_a^b A(x)\,dx = \underbrace{\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx}_{\text{ITERATED INTEGRAL}}$$

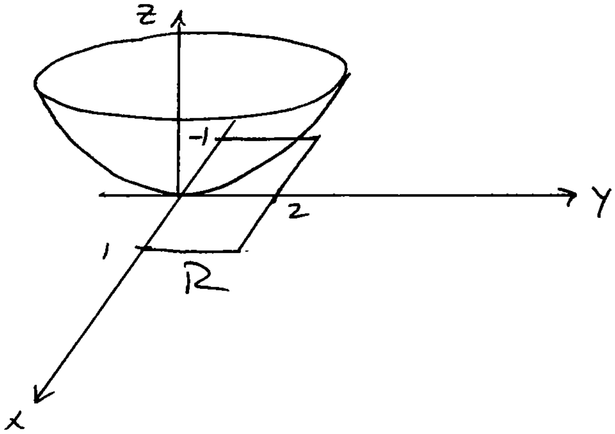
LIKEWISE WE CAN SWAP THE RÔLES OF x AND y TO OBTAIN ANOTHER EXPRESSION FOR V.



$$A(y) = \int_a^b f(x,y)\,dx$$

$$\therefore\ V = \int_c^d A(y)\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

Ex. DETERMINE THE VOLUME UNDER THE PARABOLOID $z = x^2+y^2$ AND OVER THE RECTANGLE $R = [-1,1] \times [0,2]$



$$\iint_R (x^2+y^2)\,dA = \int_0^2 \int_{-1}^1 (x^2+y^2)\,dx\,dy = \cdots = \frac{20}{3}$$

$$= \int_{-1}^1 \int_0^2 (x^2+y^2)\,dy\,dx = \cdots = \frac{20}{3}.$$

Ex. FIND $\iint_R \cos x \sin y \, dx\,dy$ WHERE $R = [0,\frac{\pi}{2}] \times [0,\frac{\pi}{2}]$.

ANS: 1

CAN WE ALWAYS CHANGE THE ORDER OF INTEGRATION, i.e. $dx\,dy$ vs. $dy\,dx$? FUBINI'S THEOREM SAYS THAT UNDER SOME MILD CONDITIONS, WE CAN. MORE ON THIS LATER.