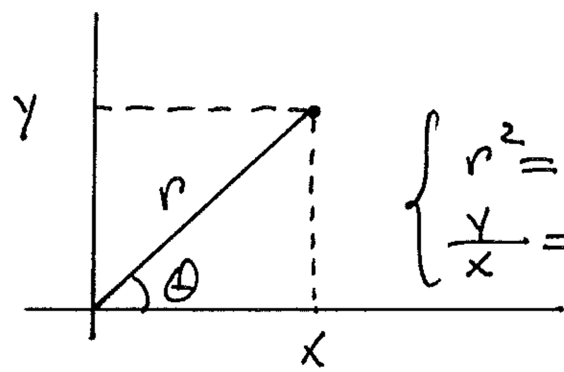
(15.4) POLAR COORDINATES

THE <u>POLAR COORDINATES</u> (r, θ) OF A POINT $(x, y) \in \mathbb{R}^2$ ARE GIVEN BY

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

i.e.



$$\begin{cases} r^2 = x^2 + y^2 \\ \dfrac{y}{x} = \tan\theta \end{cases}$$

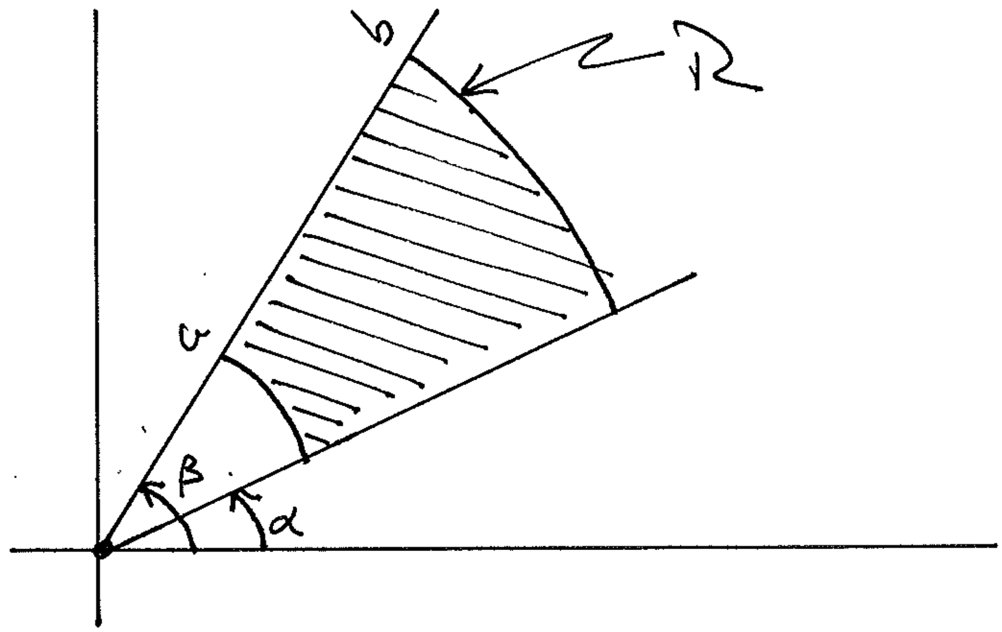
NOTE: θ is UNDEFINED FOR $r = 0$.

i.e.

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

A <u>POLAR RECTANGLE</u> is A REGION OF THE FORM

$$R = \left\{ (r, \theta) \;\middle|\; a \le r \le b \text{ and } \alpha \le \theta \le \beta \right\}$$



TO INTEGRATE OVER SUCH A REGION WE PARTITION INTO SUB-(POLAR) RECTANGLES

$$\underset{a}{\overset{\parallel}{r_0}} < r_1 < r_2 < \dots < r_{m-1} < \underset{b}{\overset{\parallel}{r_m}}$$

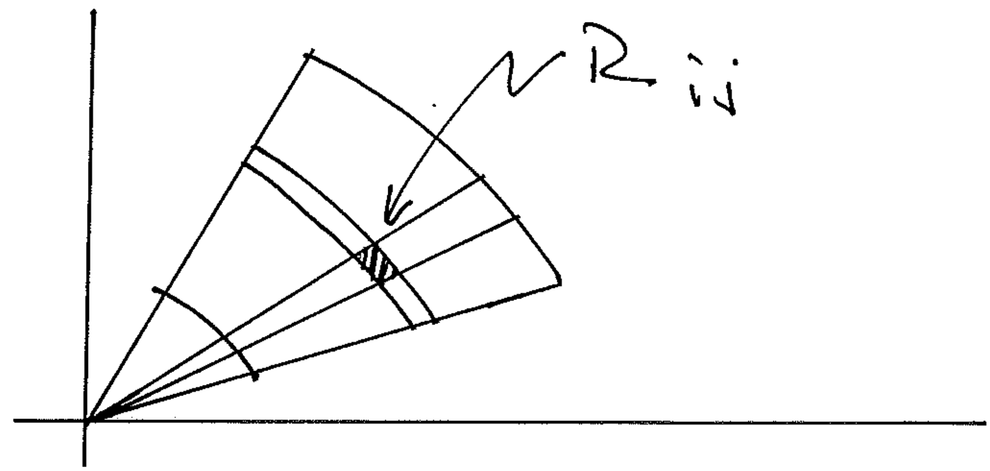
$$\underset{\alpha}{\overset{\parallel}{\theta_0}} < \theta_1 < \theta_2 < \dots < \theta_{m-1} < \underset{\beta}{\overset{\parallel}{\theta_m}}$$

with $\Delta r = r_i - r_{i-1} = \dfrac{b-a}{m} \quad (1 \le i \le m)$,

$\Delta\theta = \theta_j - \theta_{j-1} = \dfrac{\beta - \alpha}{n} \quad (1 \le j \le n)$

and

$$R_{ij} = \{(r,\theta) \mid r_{i-1} \le r < r_i \text{ and } \theta_{j-1} \le \theta \le \theta_j\}$$



WE CHOOSE SAMPLE POINTS $(r_i^*, \theta_j^*) \in R_{ij}$ TO BE MIDPOINTS

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \qquad (1 \le i \le m)$$

$$\theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j) \qquad (1 \le j \le n)$$

THE AREA OF R_{ij} IS THEN

$$\Delta A_i = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta$$

$$= \frac{1}{2}(r_i^2 - r_{i-1})^2 \Delta\theta$$

$$= \frac{1}{2}(r_{i-1} + r_i)(r_i - r_{i-1}) \Delta\theta$$

$$= r_i^* \Delta r \Delta\theta$$

NOTICE THAT THE AREA OF R_{ij} DEPENDS ON i (BUT NOT j).

THE DOUBLE INTEGRAL $\iint_R f\,dA$ is APPROXIMATED BY THE RIEMANN SUM

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^* \,\Delta r\,\Delta\theta$$

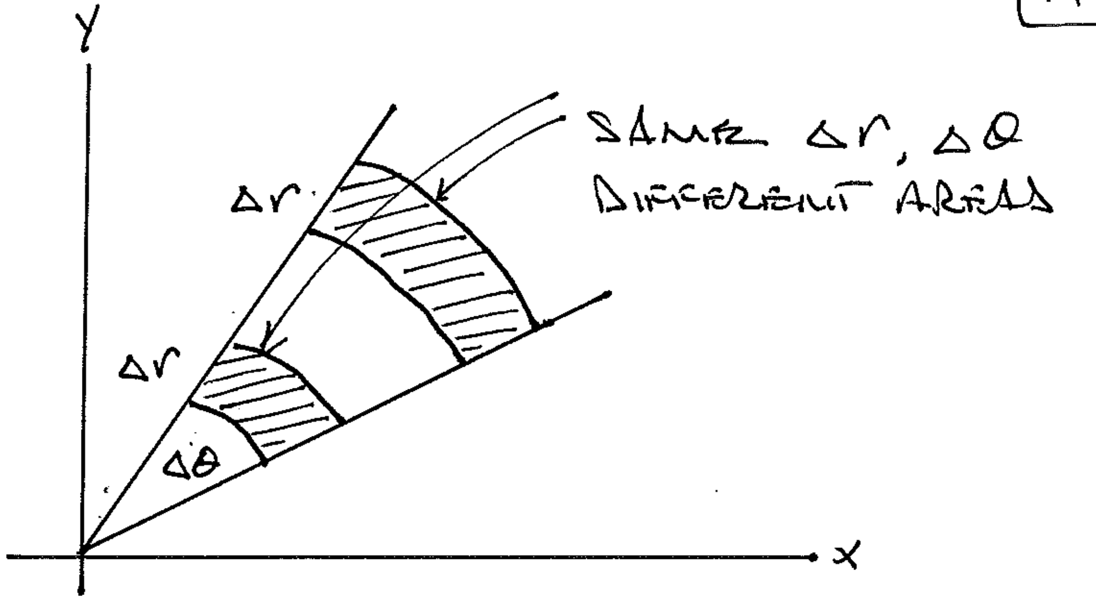
TAKING THE LIMIT OF THESE SUMS AS $m, n \to \infty$ GIVES

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

THE 'AREA FORM' dA is THUS GIVEN BY

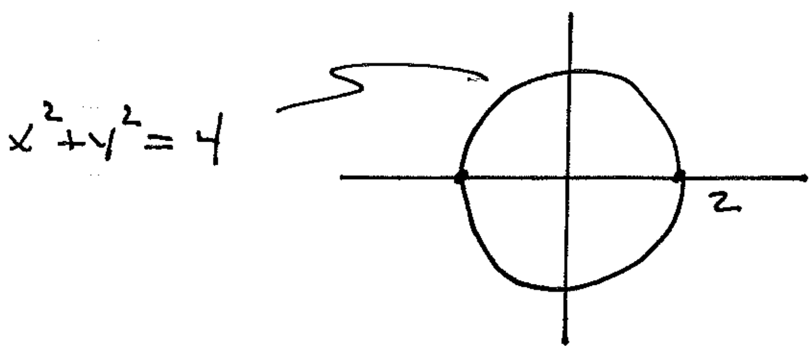
$$dA = \underbrace{dx\,dy = dy\,dx}_{\text{RECTANGULAR COORDINATES}} = \underbrace{r\,dr\,d\theta}_{\text{POLAR COORDINATES}}$$

THE FACTOR 'r' is PRESENT BECAUSE THE AREA OF A POLAR RECTANGLE DEPENDS ON ITS DISTANCE FROM THE ORIGIN.



<u>EX</u> $D = \{(x,y) \mid x^2+y^2 \le 4\}$.

EVALUATE $\displaystyle\iint_D e^{-x^2-y^2}\,dA$.



$$\begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\iint_D e^{-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^2 e^{-r^2}\, r\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi} e^{-r^2}\Big|_0^2 \, d\theta$$

$$= \left(-\tfrac{1}{2}\right) \int_0^{2\pi} \left(e^{-4} - 1\right) d\theta$$

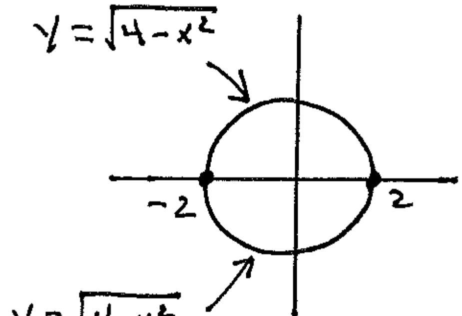
$$= \left(\frac{1 - e^{-4}}{2}\right) \theta \Big|_0^{2\pi} = \frac{1 - e^{-4}}{\cancel{2}} \cdot \cancel{2}\pi$$

$$= \boxed{\pi\left(1 - e^{-4}\right)}$$

HOW WOULD WE SET THIS UP IN RECTANGULAR COORDINATES ?

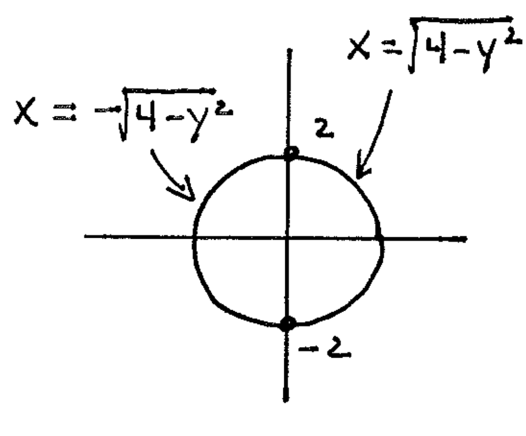
TYPE I :

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-x^2-y^2} \, dy \, dx$$



TYPE II :

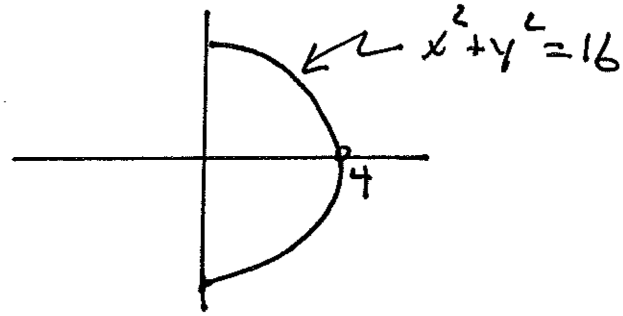
$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-x^2-y^2} \, dx \, dy$$



NEITHER INTEGRAL IS FEASIBLE IN THESE COORDINATES .

EX. $D = \{(x,y) \mid x^2+y^2 \le 16 \text{ and } x \ge 0\}$

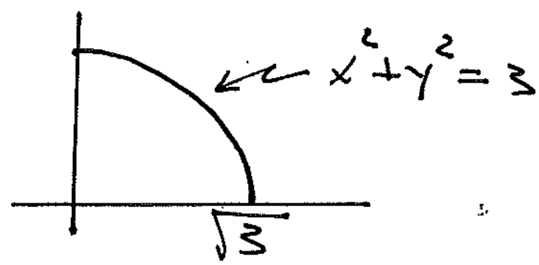
find $\iint\limits_D \sqrt{16-x^2-y^2}\, dA$



ANSWER : $\frac{64\pi}{3}$.

EX. $D = \{(x,y) \mid x^2+y^2 \le 3,\ x \ge 0,\ y \ge 0\}$

find $\iint\limits_D xy\, dA$



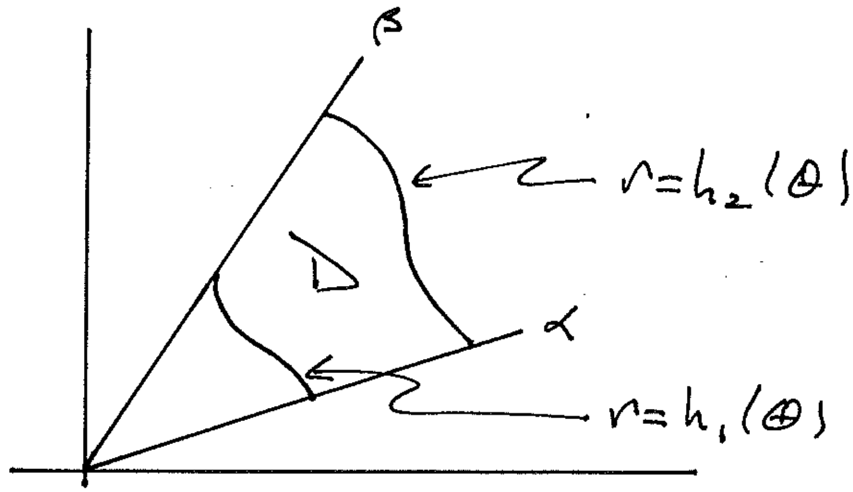
ANSWER : $\frac{9}{8}$.

EXERCISE : SET UP BOTH THESE EXAMPLES IN RECTANGULAR COORDINATES. BOTH ARE DO-ABLE, BUT NOT AS SIMPLE AS POLAR COORDINATES.

WE CAN USE POLAR COORDINATES TO INTEGRATE OVER REGIONS THAT ARE NOT POLAR RECTANGLES, BUT WHICH ARE EASILY DESCRIBED IN POLAR COORDINATES.

FOR INSTANCE, IF f IS CONTINUOUS ON

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta \text{ and } h_1(\theta) \le r \le h_2(\theta)\}$$



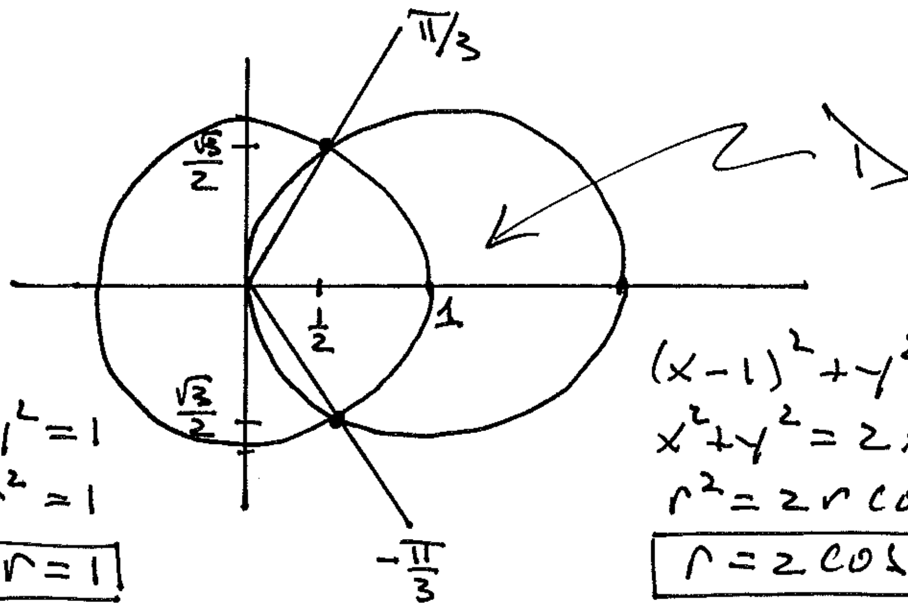
THEN

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

THIS IS THE POLAR EQUIVALENT OF A TYPE I REGION.

EX.

DETERMINE THE AREA LYING WITHIN THE CIRCLE $(x-1)^2+y^2=1$ AND OUTSIDE THE CIRCLE $x^2+y^2=1$.



$x^2+y^2=1$

$r^2=1$

$\boxed{r=1}$

$(x-1)^2+y^2=1$

$x^2+y^2=2x$

$r^2=2r\cos\theta$

$\boxed{r=2\cos\theta}$

$$\text{AREA}=\iint_D 1\,dA=\int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} r\,dr\,d\theta$$

$$=\frac{1}{2}\int_{-\pi/3}^{\pi/3} r^2\Big|_{1}^{2\cos\theta}\,d\theta$$

$$=\frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(4\cos^2\theta-1\right)d\theta$$

$$= \frac{1}{2}\left(4\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right) - \theta\right)\Bigg|_{-\pi/3}^{\pi/3}$$

(BY FORMULA #64)

$$= \frac{1}{2}\left(\theta + \sin 2\theta\right)\Bigg|_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

$$= \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}$$