(15.1) DOUBLE INTEGRALS

RECALL THE DEFINITION OF THE INTEGRAL OF $f(x)$ ON $[a,b]$:

PARTITION $[a,b]$ INTO n SUBINTERVALS OF EQUAL WIDTHS:

$$\Delta x = \frac{b-a}{n}$$

$$[\underset{\overset{\shortparallel}{a}}{x_0}, x_1], [x_1, x_2], \ldots, [x_{n-1}, \underset{\overset{\shortparallel}{b}}{x_n}]$$

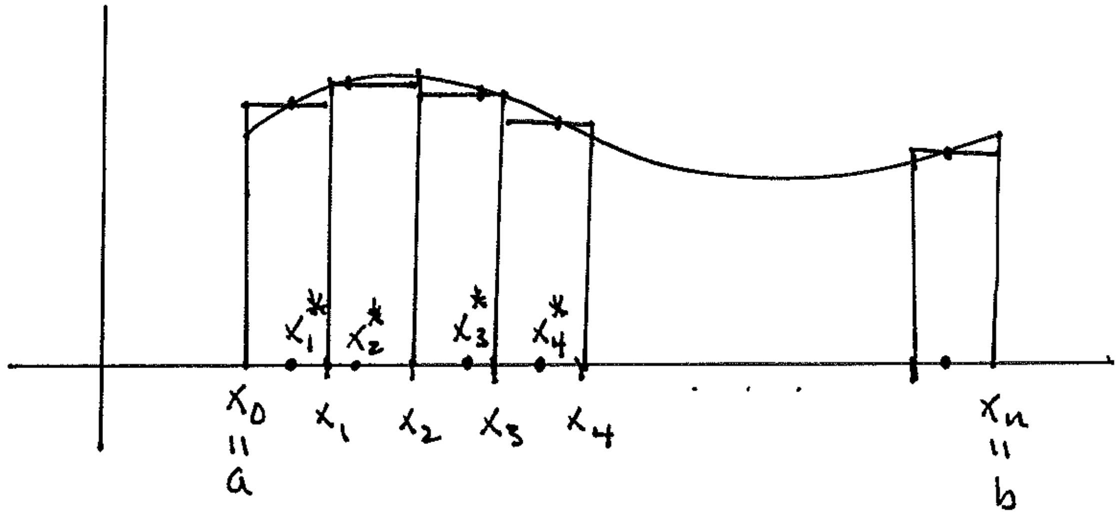
CHOOSE A POINT IN THE i^{TH} SUB-INTERVAL

$$x_i^* \in [x_{i-1}, x_i]$$

AND FORM THE RIEMANN SUM

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

AS AN APPROXIMATION TO THE (SIGNED) AREA BETWEEN THE GRAPH $y = f(x)$ AND THE x AXIS.



THEN THE DEFINITE INTEGRAL OF f OVER $[a,b]$ IS THE limit AS $n \to \infty$ OF SUCH RIEMANN SUMS

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

NOW LET $f(x,y)$ BE A FUNCTION OF 2 VARIABLES WHOSE DOMAIN INCLUDES THE CLOSED RECTANGLE

$$R = [a,b] \times [c,d]$$
$$= \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b \text{ and } c \le y \le d\}$$

TO BEGIN WE SUPPOSE $f(x,y) \ge 0$ FOR $(x,y) \in R$.

LET S BE THE SOLID REGION IN $\mathbb{R}^3$ LYING ABOVE R AND BELOW THE GRAPH $z = f(x,y)$.

GOAL: FIND THE VOLUME V OF S

WE PARTITION BOTH INTERVALS $[a,b]$ AND $[c,d]$ AS FOLLOWS:

$$[\underset{\overset{\shortparallel}{a}}{x_0}, x_1], [x_1, x_2], \ldots, [x_{m-1}, \underset{\overset{\shortparallel}{b}}{x_m}] \qquad \Delta x = \frac{b-a}{m}$$

$$[\underset{\overset{\shortparallel}{c}}{y_0}, y_1], [y_1, y_2], \ldots, [y_{n-1}, \underset{\overset{\shortparallel}{d}}{y_n}] \qquad \Delta y = \frac{d-c}{n}$$

THIS EFFECTIVELY PARTITIONS R INTO $m \cdot n$ SUBRECTANGLES:

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$
$$= \{(x,y) \mid x_{i-1} \le x \le x_i \text{ and } y_{j-1} \le y \le y_j\}$$

for $1 \le i \le m$ AND $1 \le j \le n$, EACH WITH AREA

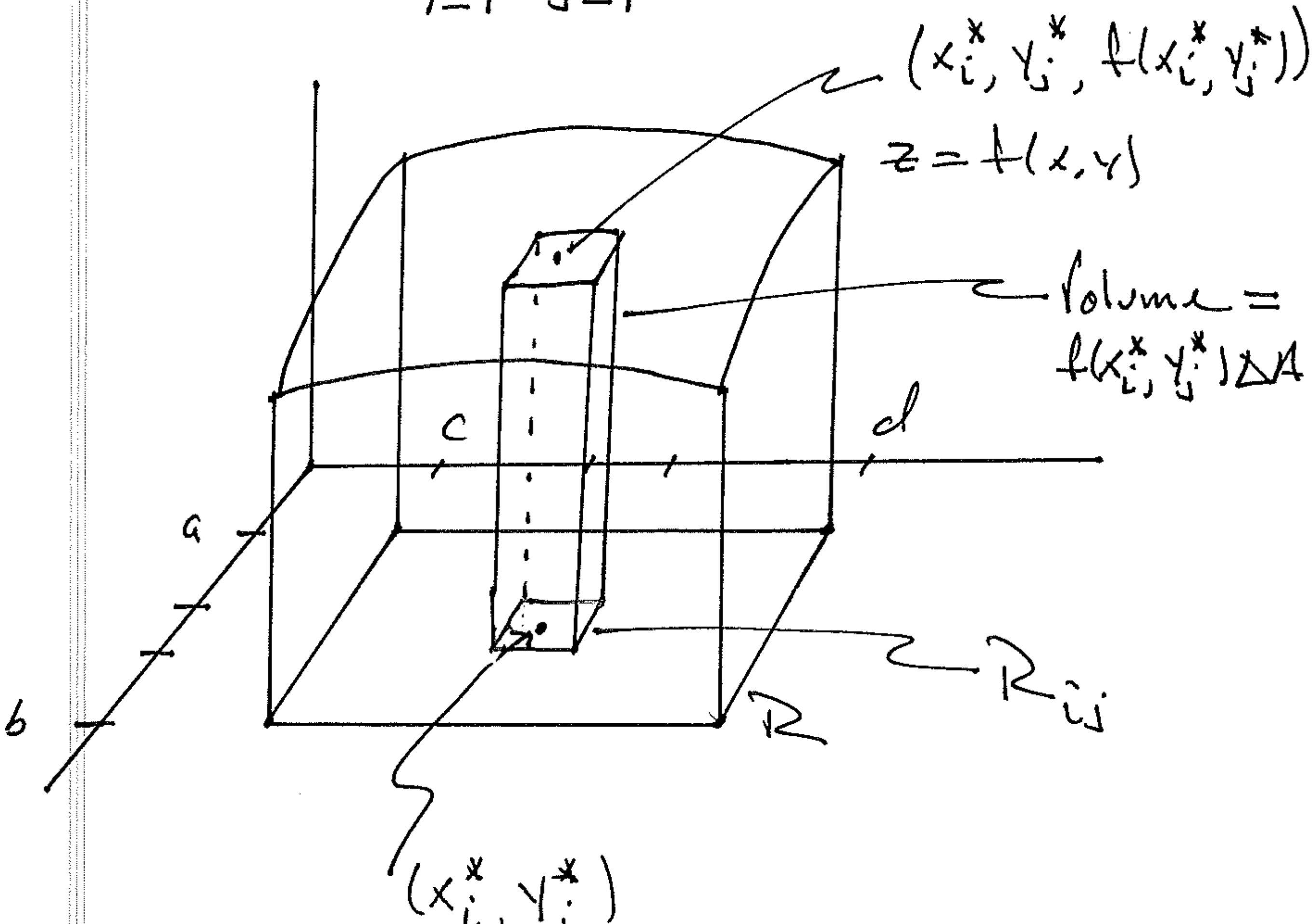
$$\Delta A = \Delta x \cdot \Delta y$$

WE CHOOSE SAMPLE POINTS

$$(x_i^*, y_i^*) \in R_{ij}$$

AND FORM THE CORRESPONDING RIEMANN SUM AS AN APPROXIMATION TO V :

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A$$



THE ACTUAL VALUE OF V IS OBTAINED BY TAKING THE limit AS $n \to \infty$ AND $m \to \infty$.

i.e.
$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$
PROVIDED THIS LIMIT EXISTS.

DEFN:
THE DOUBLE INTEGRAL OF $f(x,y)$ OVER R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A$$

if THIS LIMIT EXISTS.

NOTE: WE CAN FORM THE DOUBLE INTEGRAL WHETHER OR NOT $f(x,y) \geq 0$ ON R. IN THE LATTER CASE

$$\iint_R f(x,y)\,dA$$

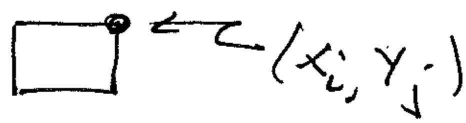
is THE SIGNED VOLUME lying BETWEEN R AND THE GRAPH $z = f(x,y)$.

IT IS REQUIRED IN THE ABOVE DEFINITION THAT THE limit EXIST INDEPENDENTLY OF THE CHOICE OF SAMPLE POINTS

$$(x_i^*, y_i^*) \in R_{ij}$$

SOMETIMES A COHERENT CHOICE OF (x_i^*, y_j^*) IS HELPFUL, SUCH AS

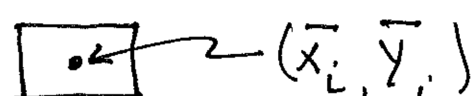
$$\left.\begin{aligned} x_i^* &= x_i \\ y_j^* &= y_j \end{aligned}\right\} \text{UPPER RIGHT CORNER OF } R_{ij}$$



ANOTHER POSSIBLE CHOICE IS THE MIDPOINT:

$$x_i^* = \bar{x}_i = \frac{x_{i-1} + x_i}{2}$$

$$y_j^* = \bar{y}_j = \frac{y_{j-1} + y_j}{2}$$



IN ANY CASE, THE RESULTING RIEMANN SUM IS A GOOD APPROXIMATION TO THE DOUBLE INTEGRAL.

EX LET $R = [1,2] \times [1,3]$ AND $f(x,y) = 2x + 3y$.

a.) APPROXIMATE $\iint_R (2x+3y)\,dA$ BY TAKING $x_i^* = x_i$, $y_j^* = y_j$, $m=10$, $n=20$.

b.) FIND THE EXACT VALUE OF $\iint_R (2x+3y)\,dA$ BY TAKING $x_i^* = x_i$, $y_j^* = y_j$ AND TAKING limit $n \to \infty$, $m \to \infty$.

SOLUTION:

WE HAVE

$$x_i^* = x_i = 1 + \frac{i}{m} \qquad (1 \le i \le m)$$

$$y_j^* = y_j = 1 + \frac{2j}{n} \qquad (1 \le j \le n)$$

$$\Delta x = \frac{2-1}{m} = \frac{1}{m}, \quad \Delta y = \frac{3-1}{n} = \frac{2}{n}$$

$$\therefore \quad \Delta A = \Delta x \, \Delta y = \frac{2}{mn}$$

THUS

$$\iint_R (2x+3y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\left(5+\frac{2i}{m}+\frac{6j}{n}\right)\cdot\frac{2}{m\cdot n}$$

TAKING $m=10$, $n=20$ WE HAVE

$$\iint_R (2x+3y)\,dA \approx \boxed{18.5}$$

FOR THE GENERAL CASE, WE USE

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2} \quad \text{AND} \quad \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

TO OBTAIN

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(5+\frac{2i}{m}+\frac{6j}{n}\right)\cdot\frac{2}{mn} = 18+\frac{2}{m}+\frac{6}{n}$$

WHENCE

$$\iint_R (2x+3y)\,dA = \lim_{m,n\to 0}\left(18+\frac{2}{m}+\frac{6}{n}\right) = \boxed{18}$$

<u>EXERCISE</u>: RE-DO PART (a) WITH

$$x_i^* = \bar{x}_i = \frac{x_{i-1}+x_i}{2}, \quad y_j^* = \bar{y}_j = \frac{y_{j-1}+y_j}{2}$$

$m=10$, $n=20$, AND LIKEWISE PART (b).

Defn

The average value of $f(x,y)$ over a rectangle $R \subseteq \mathbb{R}^2$ is

$$f_{avg.} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

Thus if $f(x,y) \geq 0$ on R, the equation

$$f_{avg} \cdot A(R) = \iint_R f(x,y)\, dA$$

says that f_{avg} is the height of the rectangular box with base R having volume equal to the solid region under the graph $z = f(x,y)$ and above R.

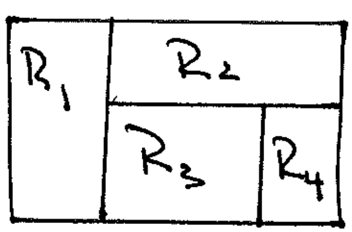
Double integrals have the standard properties one would expect i.e.

- $$\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA$$

- $$\iint_R cf\,dA = c\iint_R f\,dA \quad (\text{any } c \in \mathbb{R}).$$

- If $f(x,y) \geq g(x,y)$ for $(x,y) \in R$, THEN
$$\iint_R f\,dA \geq \iint_R g\,dA$$

- If R is DECOMPOSED INTO SUB-RECTANGLES $R_1, \ldots, R_K$ THEN
$$\iint_R f\,dA = \iint_{R_1} f\,dA + \ldots + \iint_{R_K} f\,dA$$

e.g. R 

ALL THESE PROPERTIES FOLLOW FROM STANDARD FACTS CONCERNING LIMITS AND SUMMATIONS.