(14.6) DIRECTIONAL DERIVATIVES

DEFN LET $\vec{u} = \langle a, b \rangle$ BE A UNIT VECTOR IN $\mathbb{R}^2$ AND $f(x,y)$ A FCN OF 2 VAR. THE DIRECTIONAL DERIVATIVE OF f AT (x_0, y_0) IN THE DIRECTION $\vec{u}$ IS

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

1ST OBSERVE THAT:

$$D_{\vec{u}} f(x_0, y_0) = \left. \frac{d}{dt} \right|_{t=0} f(x_0 + at, y_0 + bt)$$

By chain Rule:

$$D_u f(x_0, y_0) = f_x(x_0, y_0) \cdot a + f_y(x_0, y_0) \cdot b$$

ANY OF THE ABOVE FORMULI COULD BE THE DEFN. OF $D_{\vec{u}} f(x_0, y_0)$.

NOTE TWO SPECIAL CASES: $\vec{u} = \vec{i} = \langle 1, 0 \rangle$

$$D_{\vec{i}} f(x_0, y_0) = f_x(x_0, y_0)$$

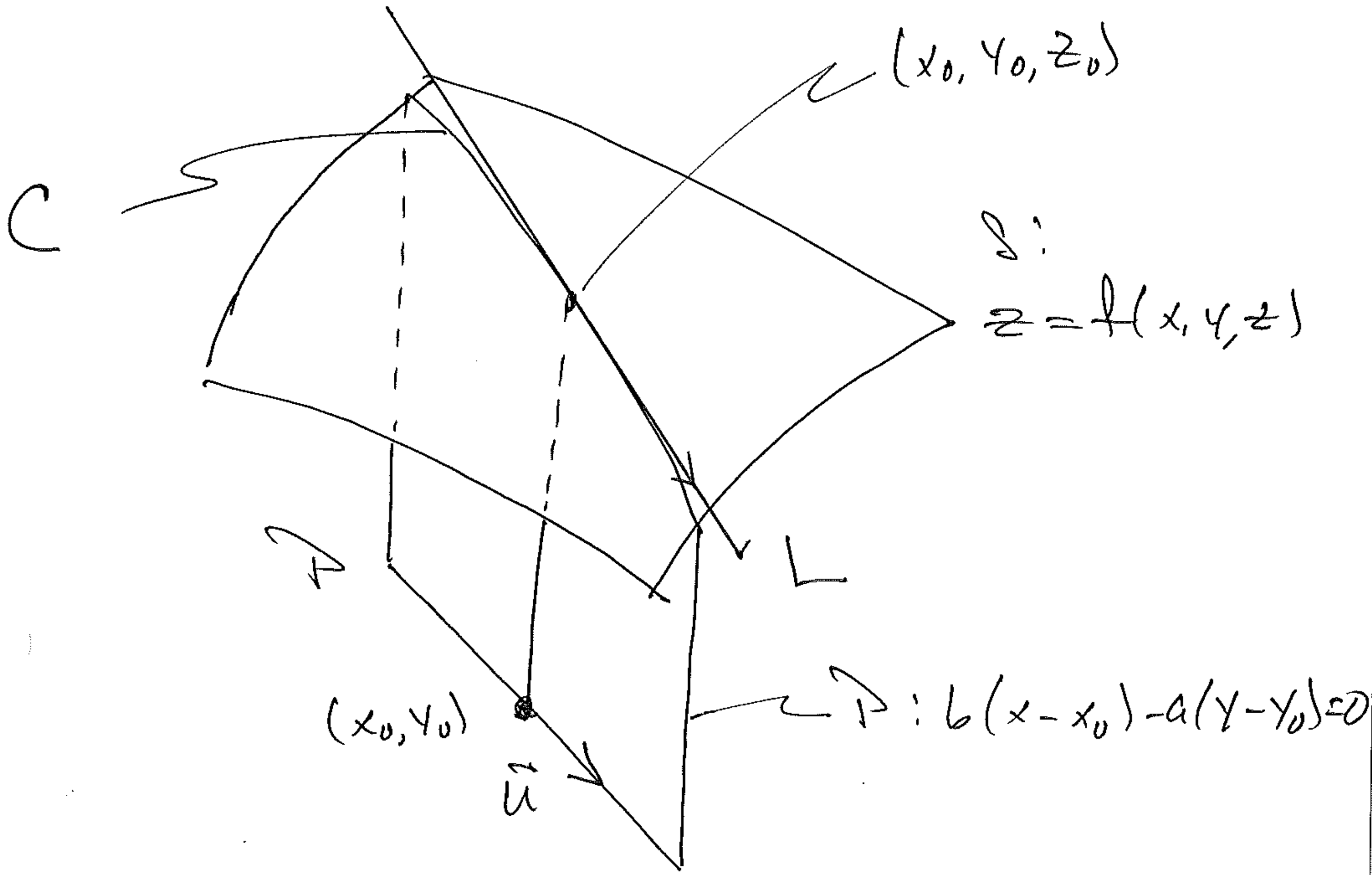
AND: $\vec{u} = \vec{j} = \langle 0, 1 \rangle$

$$D_{\vec{j}} f(x_0, y_0) = f_y(x_0, y_0)$$

WE SEE THAT $D_{\vec{u}}$ IS A GENERALIZATION OF PARTIAL DIFFERENTIATION

AS BEFORE LET S BE THE SURFACE $z = f(x, y)$, AND NOW LET C BE THE INTERSECTION OF S WITH THE PLANE P $\perp$ TO xy-PLANE THROUGH (x_0, y_0) AND PARALLEL TO $\vec{u}$. LET L BE TANGENT LINE TO C AT (x_0, y_0, z_0) WHERE $z = f(x_0, y_0)$.

THEN $D_u f(x_0, y_0)$ IS THE SLOPE OF L IN THE PLANE P.



C IS PARAM. BY $\vec{r}(t) = \langle x_0 + at, y_0 + bt, f(x_0+at, y_0+bt) \rangle$

$\therefore \vec{r}(0) = \langle x_0, y_0, z_0 \rangle$

$\therefore \vec{r}\,'(t) = \langle a, b, \frac{d}{dt} f(x_0+at, y_0+bt) \rangle$

$$\vec{r}\,'(0) = \langle a, b, \frac{d}{dt}\Big|_{t=0} f(x_0+at, y_0+bt) \rangle$$

$$= \langle a, b, D_{\vec{u}} f(x_0, y_0) \rangle$$

THUS $D_{\vec{u}} f(x_0, y_0)$ IS THE RATE OF CHANGE IN THE Z-COORDINATE OF A POINT ON S AS (x, y) TRAVELS AT UNIT SPEED ALONG THE DIRECTION $\vec{u}$.

<u>NOTE</u>: IT IS ESSENTIAL THAT $\vec{u}$ BE A UNIT VECTOR IN THIS INTERPRETATION.

OBSERVE THAT for $\vec{u} = \langle a, b \rangle$

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$
$$= \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle a, b \rangle.$$

<u>DEFN</u>: THE <u>GRADIENT VECTOR</u> OF $f(x, y)$ is

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

THUS

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

OR BRIEFLY

$$\boxed{D_{\vec{u}} f = \nabla f \cdot \vec{u}} = \text{comp}_{\vec{u}} \nabla f$$

IN SOME TEXTS THIS IS TAKEN AS DEFN OF DIRECTIONAL DERIVATIVE.

Ex. $f(x,y) = x^2y^3 - 3xy^4$

$$\nabla f = \langle 2xy^3 - 3y^4, \; 3x^2y^2 - 12xy^3 \rangle$$

DIR DIR OF f AT $(1,1)$ IN DIR $\vec{u} = \left\langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \right\rangle$:

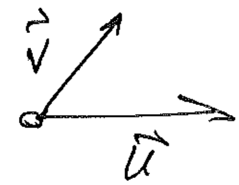
$$D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \vec{u}$$

$$= \langle -1, -9 \rangle \cdot \left\langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \right\rangle$$

$$= \frac{-3+18}{\sqrt{13}} = \boxed{\frac{15}{\sqrt{13}}}$$

RECALL FOR ANY $\vec{u}, \vec{v} \in \mathbb{R}^2$:

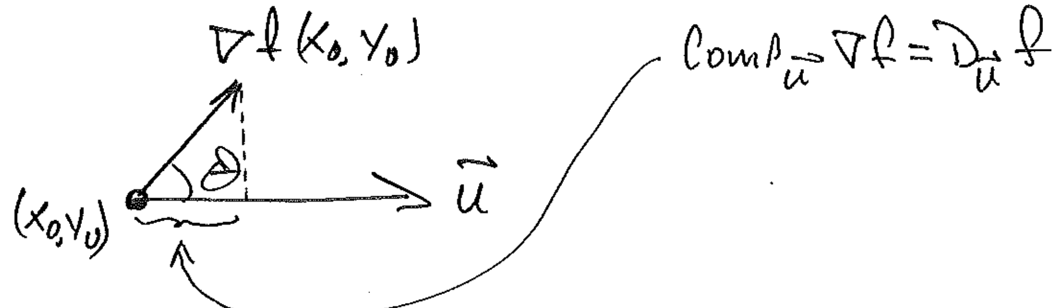
$$\vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}| \cos\theta$$



IF $\vec{v} = \nabla f$, AND $\vec{u}$ IS A UNIT VECTOR THEN

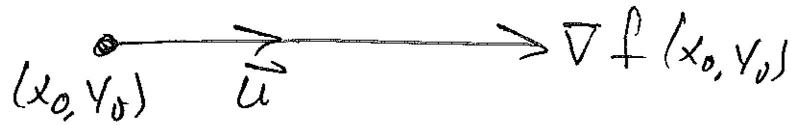
$$D_{\vec{u}} f = |\nabla f| \cos\theta$$

WHERE θ IS THE ANGLE BETWEEN ∇f AND $\vec{u}$



FOR WHICH DIRECTION $\vec{u}$ IS $D_{\vec{u}} f(x_0, y_0)$ THE GREATEST? LEAST?

max: $\theta = 0 \Rightarrow \cos\theta = 1$: $D_{\vec{u}} f(x_0, y_0) = |\nabla f(x_0, y_0)|$



min: $\theta = -1 \Rightarrow \cos\theta = -1$: $D_{\vec{u}} f(x_0, y_0) = -|\nabla f(x_0, y_0)|$

THUS $\nabla f(x_0, y_0)$ POINTS IN THE DIR. OF MAXIMUM INCREASE OF f AND $-\nabla f(x_0, y_0)$ IS THE DIRECTION OF MAXIMUM DECREASE. THE RATE OF INCREASE/DECREASE IS IN EACH CASE $|\nabla f(x_0, y_0)|$.

WE GENERALIZE ALL OF THIS TO FCNS OF 3, 4, ..., n VARIABLES.

LET $\vec{u} = \langle a, b, c \rangle$ BE A UNIT VECTOR IN $\mathbb{R}^3$

AND $f(x, y, z)$ A FCN. OF 3 VARS.

$$D_{\vec{u}} f(x,y,z) = \lim_{h \to 0} \frac{f(x+ah, y+bh, z+ch) - f(x,y,z)}{h}$$

$$= \frac{d}{dt}\Big|_{t=0} f(x+at, y+bt, z+ct)$$

$$= f_x a + f_y b + f_z c$$

$$= \langle f_x, f_y, f_z \rangle \cdot \langle a, b, c \rangle$$

$$= \nabla f \cdot \vec{u} = |\nabla f| \cos\theta = \mathrm{Comp}_{\vec{u}} \nabla f$$

WHERE $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$.

<u>NOTE</u>: ∇f is AN EXAMPLE OF A <u>VECTOR FIELD</u> i.e. A VECTOR ATTACHED TO EACH POINT IN SPACE

EX. FIND DIR. DER. OF

$$f(x,y,z) = \sqrt{xyz}$$

AT $(1,4,1)$ IN DIR OF $\vec{v} = \langle 2, 3, 6 \rangle$.

$$\nabla f = \left\langle \frac{yz}{2\sqrt{xyz}}, \frac{xz}{2\sqrt{xyz}}, \frac{xy}{2\sqrt{xyz}} \right\rangle$$

$$\nabla f(1,4,1) = \left\langle \frac{4}{2 \cdot 2}, \frac{1}{2 \cdot 2}, \frac{4}{2 \cdot 2} \right\rangle = \left\langle 1, \frac{1}{4}, 1 \right\rangle$$

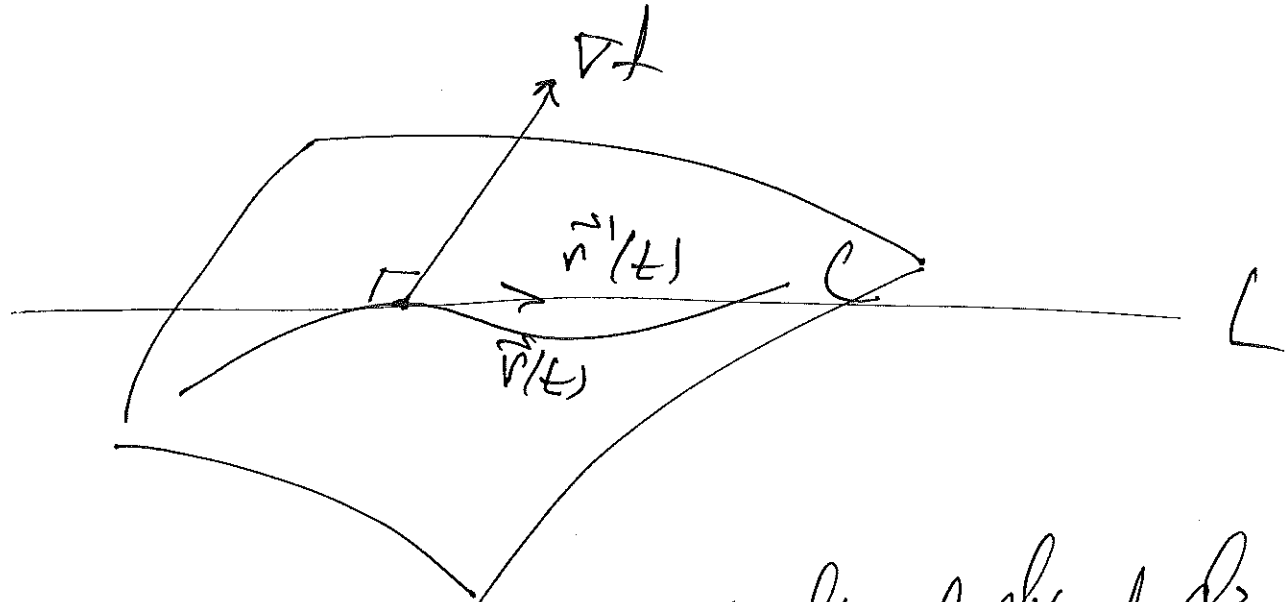
$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$$

$$D_{\vec{u}} f(1,4,1) = \left\langle 1, \frac{1}{4}, 1 \right\rangle \cdot \left\langle \frac{2}{7}, \frac{3}{7}, \frac{1}{7} \right\rangle$$

$$= \frac{2}{7} + \frac{3}{28} + \frac{6}{7} = \frac{8+3+24}{28} = \boxed{\frac{35}{28}}$$

LET S BE A LEVEL SURFACE OF f, i.e. $f(x,y,z) = \text{const}$ AND LET C BE ANY CURVE LYING IN S. SAY C IS PARAM. BY $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ SO

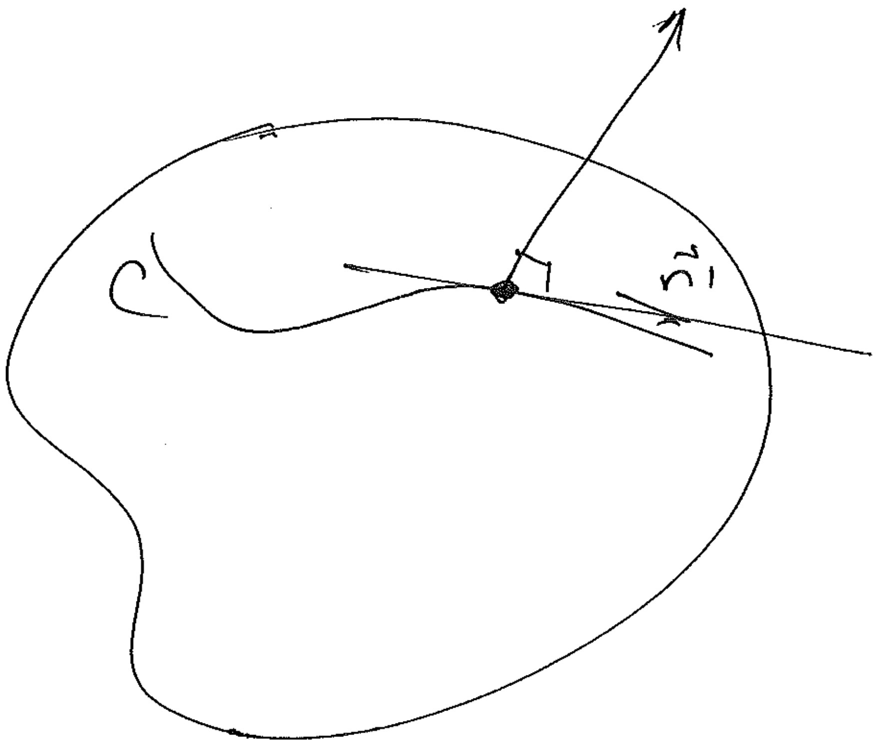
$$f(x(t), y(t), z(t)) = \text{const}$$



$$\therefore\ 0 = \frac{d}{dt} f(x(t), y(t), z(t)) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

$= \nabla f \cdot \vec{r}\,'$ THUS ∇f IS $\perp$ TO C

$\therefore \nabla f$ is $\perp$ to S

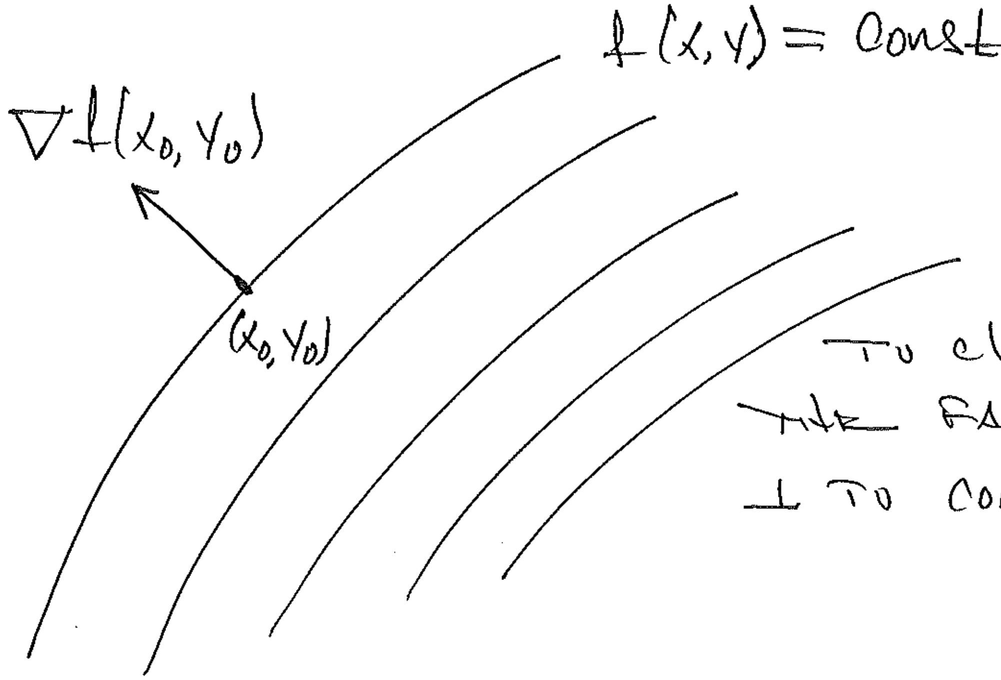


$\therefore \nabla f(x_0, y_0, z_0)$ is normal to S at (x_0, y_0, z_0). Thus

tangent plane is:

$$\boxed{f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0}$$

In the 2-dim case: ∇f is $\perp$ to the level curves of $f(x,y)$:



To climb hill the fastest, walk $\perp$ to contour lines.

<u>Ex.</u> Find eqns. of tangent plane & normal line to S at P.

$$S: y = x^2 - z^2$$

$$P(4, 7, 3)$$

$$f(x,y,z) = y - x^2 + z^2 = 0$$

$$\nabla f = \langle -2x, 1, 2z \rangle$$

$\vec{n} = \langle -8, 1, 6 \rangle$

PLANE: $-8(x-4)+(y-7)+6(z-3)=0$

NORMAL LINE:
PARA:
$$\begin{cases} x = 4 - 8t \\ y = 7 + t \\ z = 3 + 6t \end{cases}$$

SYMM: $\dfrac{x-4}{-8} = \dfrac{y-7}{1} = \dfrac{z-3}{6}$

EX. AT WHAT POINT ON $y = x^2 + z^2$ IS THE TANGENT PLANE PARALLEL TO $x+2y+3z=1$.

ANS. AT $\left(-\frac{1}{4}, \frac{10}{16}, -\frac{3}{4}\right)$