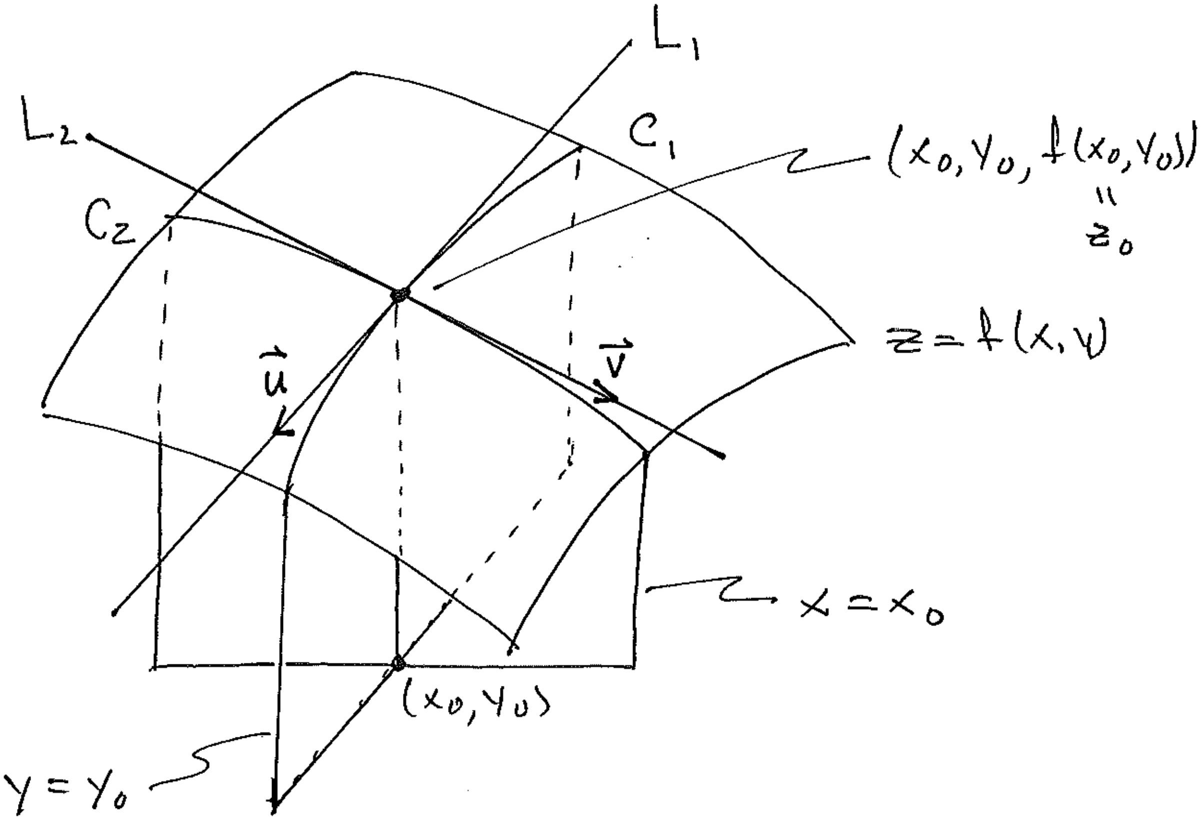
# (14.4) TANGENT PLANES & LINEAR APPROX.

LET $f(x,y)$ BE A FUNCTION OF 2 VARIABLES, $S$ ITS GRAPH $z=f(x,y)$, AND AS BEFORE LET $C_1, C_2$ BE THE CURVES OBTAINED BY INTERSECTING $S$ WITH THE VERTICAL PLANES $y=y_0$ AND $x=x_0$, RESPECTIVELY.




LET $L_1$ AND $L_2$ BE THE TANGENT LINES TO $C_1$ AND $C_2$ (RESP.) AT THE POINT $(x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$.

**DEFN:** THE TANGENT PLANE TO $S$ AT $(x_0, y_0, z_0)$ IS THE PLANE CONTAINING THE LINES $L_1$ AND $L_2$.

LET $\vec{u}, \vec{v}$ BE DIRECTION VECTORS FOR $L_1$ AND $L_2$ RESPECTIVELY. THEN $\vec{n} = \vec{u} \times \vec{v}$ IS A NORMAL VECTOR TO THE TANGENT PLANE TO $S$ AT $(x_0, y_0, z_0)$.

TO FIND $\vec{u}$ WE PARAMETERIZE $C_1$ BY

$$\vec{r}(t) = \langle t, y_0, f(t, y_0) \rangle$$

SO

$$\vec{r}(x_0) = \langle x_0, y_0, f(x_0, y_0) \rangle$$

THUS

$$\vec{r}\,'(t) = \langle 1, 0, f_x(t, y_0) \rangle$$

AND

$$\therefore \quad \vec{u} = \vec{r}\,'(x_0) = \langle 1, 0, f_x(x_0, y_0) \rangle$$

IN A SIMILAR MANNER WE FIND

$$\vec{v} = \langle 0, 1, f_y(x_0, y_0) \rangle$$

<u>EXERCISE</u>: CARRY OUT THESE DETAILS

THEREFORE

$$\vec{n} = \vec{u} \times \vec{v} = \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \rangle$$

NOTICE: $\vec{n}$ IS THE 'UPWARD' POINTING NORMAL.

THE TANGENT PLANE TO S AT $(x_0, y_0, z_0)$ HAS EQUATION

$$-f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + (z-z_0) = 0$$

i.e. $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

i.e. $\boxed{z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$

Ex. $z = 4 - x^2 - y^2$ AT $(1, 1, 2)$

$f(x,y) = 4 - x^2 - y^2$

$f_x = -2x$ , $f_y = -2y$

$f_x(1,1) = -2$ , $f_y(1,1) = -2$

TANGENT PLANE:

$$2(x-1) + 2(y-1) + (z-2) = 0$$

$$\boxed{2x + 2y + z = 6}$$

Ex. $z = f(x,y) = y\cos(x-y)$ AT $(2, 2, 2)$

$f_x = -y\sin(x-y)$ $\quad f_y = y\sin(x-y) + \cos(x-y)$

$f_x(2,2) = 0$ $\quad f_y(2,2) = 1$

TANGENT PLANE: $0\cdot(x-2) - (y-2) + (z-2) = 0$

$$\boxed{z = y}$$

OBSERVE THAT THE TANGENT PLANE TO A SURFACE $S$ IS 'NEAR' $S$ FOR POINTS 'NEAR' TO THE POINT OF TANGENCY.

i.e. THE LINEAR (1ST DEGREE) POLYNOMIAL

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

is A GOOD APPROXIMATION TO $f(x,y)$ for POINTS $(x,y)$ 'near' TO $(x_0,y_0)$.

DEFN
WE CALL THE FUNCTION $L(x,y)$ ABOVE THE linearization OF $f$ AT $(x_0,y_0)$. WE CALL THE APPROXIMATION FORMULA

$$f(x,y) \approx L(x,y)$$

THE linear approximation OF $f$ AT $(x_0,y_0)$. i.e.

$$f(x,y) \approx f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

for $(x,y)$ near $(x_0,y_0)$.

NOTATION:

THE INCREMENTS $\Delta x, \Delta y, \Delta z$ IN $x$, $y$, $z$ ARE

$$\Delta x = x - x_0 \qquad \text{i.e.} \quad x = x_0 + \Delta x$$
$$\Delta y = y - y_0 \qquad \text{i.e.} \quad y = y_0 + \Delta y$$

AND

$$\Delta z = f(x,y) - f(x_0,y_0) \quad \text{i.e.} \quad f(x,y) = f(x_0,y_0) + \Delta z$$

THE PRECEDING APPROXIMATION FORMULA CAN THEN BE WRITTEN

$$f(x,y) - f(x_0,y_0) \approx f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

i.e.

$$\boxed{\Delta z \approx f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y}$$

DEFN.

WE SAY $f(x,y)$ IS DIFFERENTIABLE AT $(x_0,y_0)$ IFF

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

WHERE $\varepsilon_1 \to 0$ AND $\varepsilon_2 \to 0$ AS $(\Delta x, \Delta y) \to (0,0)$.

WHY SHOULD THIS BE THE DEFINITION OF DIFFERENTIABILITY OF A FUNCTION OF 2 VARIABLES ? IN WHAT SENSE IS THIS A GENERALIZATION OF THE 1 VARIABLE CASE ?

RECALL THAT $y = g(x)$ IS DIFFERENTIABLE AT $x_0$ IFF

$$g'(x_0) = \lim_{h \to 0} \left( \frac{g(x_0+h) - g(x_0)}{h} \right) \quad \text{EXISTS.}$$

WRITE $\Delta x = h$ AND $\Delta y = g(x_0 + \Delta x) - g(x_0)$, THEN THIS LIMIT BECOMES

$$g'(x_0) = \lim_{\Delta x \to 0} \left( \frac{\Delta y}{\Delta x} \right)$$

i.e.

$$\lim_{\Delta x \to 0} \left( \frac{\Delta y}{\Delta x} - g'(x_0) \right] = 0$$

i.e.

$$\lim_{\Delta x \to} \left( \frac{\Delta y - g'(x_0)\Delta x}{\Delta x} \right) = 0$$

NOW LET $\varepsilon = \dfrac{\Delta y - g'(x_0)\Delta x}{\Delta x}$. THEN

WE SEE THAT

G IS DIFFERENTIABLE AT $x_0$ IFF

$$\Delta y = f'(x_0)\Delta x + \varepsilon \Delta x$$

WHERE $\varepsilon \to 0$ AS $\Delta x \to 0$.

THUS THE DEFINITION OF DIFFERENTIABILITY OF A FUNCTION OF 2 VARIABLES IS A DIRECT GENERALIZATION OF THE 1 VARIABLE CASE.

THEOREM
IF $f_x$ AND $f_y$ EXIST IN A DISK CONTAINING $(x_0, y_0)$ AND ARE CONTINUOUS AT $(x_0, y_0)$, THEN $f$ IS DIFFERENTIABLE AT $(x_0, y_0)$.

EX.
FIND THE LINEARIZATION TO

$$f(x,y) = \ln(x-3y)$$

AT $(7,2)$ AND USE IT TO ESTIMATE $f(6.9, 2.06)$. i.e. $x_0 = 7$, $y_0 = 2$, $\Delta x = -.1$, $\Delta y = .06$.

WE HAVE

$$f(7,2) = \ln(1) = 0$$

$$f_x = \frac{1}{x-3y} \quad \therefore f_x(7,2) = 1$$

$$f_y = \frac{-3}{x-3y} \quad \therefore f_y(7,2) = -3$$

$$\begin{aligned} \therefore L(x,y) &= 0 + 1\cdot(x-7) + (-3)(y-2) \\ &= (x-7) - 3(y-2) \\ &= \Delta x - 3\Delta y \end{aligned}$$

$$\therefore f(6.9, 2.06) \approx (-.1) - 3(.06) = \boxed{-.28}$$

NOTE THE ACTUAL VALUE IS ABOUT $f(6.9, 2.06) = -.32850\ldots$ SO THE ERROR IN THIS APPROXIMATION IS ABOUT $+.05$, i.e. $15\%$.

DEFN

THE <u>TOTAL DIFFERENTIAL</u> OF $z = f(x,y)$ IS

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

HERE $(x,y)$ IS TO BE REGARDED AS FIXED, $dx, dy$ ARE INDEPENDENT VARIABLES, AND $dz$ IS THE DEPENDENT VARIABLE.

LETTING $dx = \Delta x$, $dy = \Delta y$, WE CAN WRITE THE APPROXIMATION

$$\Delta z \approx f_x \cdot \Delta x + f_y \cdot \Delta y$$

AS

$$\Delta z \approx dz$$

NOTE :

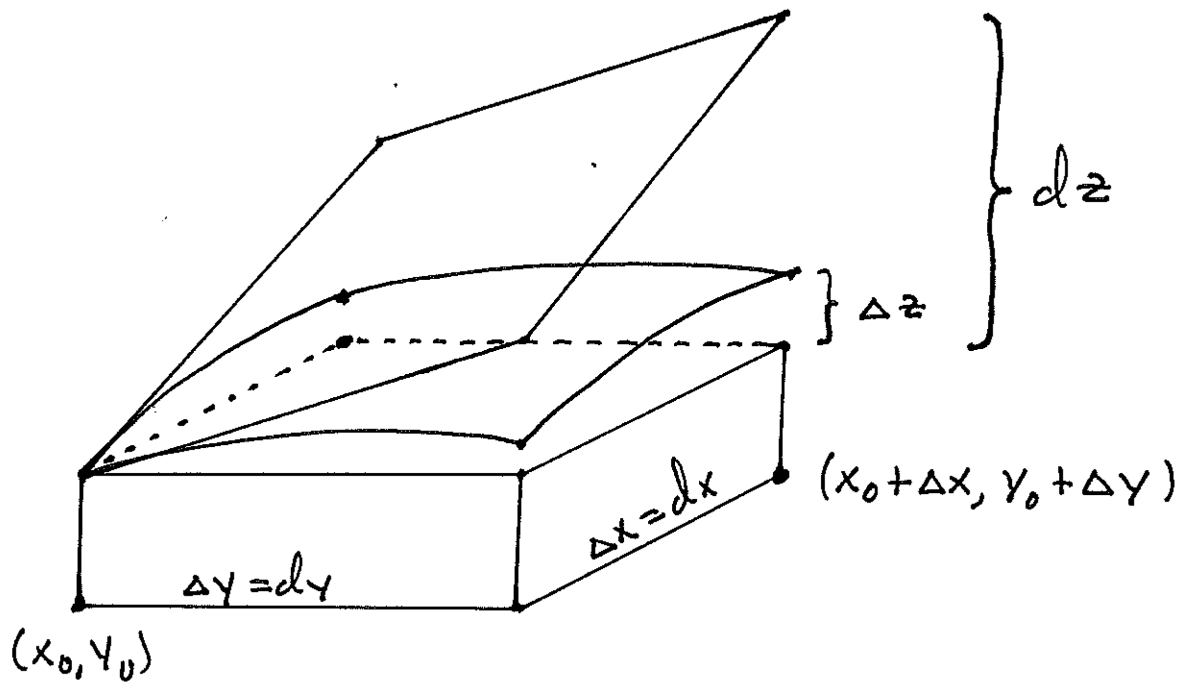
TO NEWTON AND LEIBNIZ, THE FOUNDERS OF THE CALCULUS, THE QUANTITIES $dx, dy, dz$ WERE NOT REAL NUMBERS, BUT IDEALIZATIONS CALLED 'INFINITESIMALS', WHICH WERE THOUGHT OF AS INFINITELY SMALL (NON-ZERO) NUMBERS.

OBSERVE THAT $dz$ IS THE CHANGE IN THE LINEAR APPROXIMATION $L(x,y)$ AS $(x,y)$ CHANGES FROM $(x_0, y_0)$ TO $(x_0+\Delta x, y_0+\Delta y)$, WHILE $\Delta z$ IS THE CORRESPONDING CHANGE IN $f(x,y)$

$$dz = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$$

$$= \Big(f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y\Big) - f(x_0,y_0)$$

$$= L(x_0+\Delta x, y_0+\Delta y) - L(x_0,y_0)$$




WE CAN GENERALIZE THE LINEAR APPROXIMATION AND TOTAL DIFFERENTIAL TO FUNCTIONS OF 3 OR MORE VARIABLES

e.g. $w = f(x,y,z)$, $x = x_0 + \Delta x$, $y = y_0 + \Delta y$, $z = z_0 + \Delta z$

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)\Delta x + f_y(x_0,y_0,z_0)\Delta y + f_z(x_0,y_0,z_0)\Delta z$$

AND

$$dw = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

SEE EXAMPLE 6 P. 898.