(14.3) PARTIAL DERIVATIVES

LET $f(x,y)$ BE A FUNCTION OF 2 VARIABLES, AND LET $(a,b) \in \mathrm{Dom}(f)$. CONSIDER

$$g(x) = f(x,b)$$

i.e. HOLD $y=b$ CONSTANT IN $f(x,y)$. WE CALL $g'(a)$ THE <u>PARTIAL DERIVATIVE OF f WITH RESPECT TO x AT (a,b)</u>. i.e.

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

SIMILARLY, THE <u>PARTIAL DERIVATIVE OF f WITH RESPECT TO y AT (a,b)</u> IS

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

IN EACH CASE WE HOLD ONE VARIABLE CONSTANT WHILE DIFFERENTIATING WITH RESPECT TO THE OTHER VARIABLE.

Ex. $f(x,y) = x^4 + 5x^3y^2 - 12y$

$$f_x(1,2) = \frac{d}{dx}\left[f(x,2)\right]\Big|_{x=1}$$

$$= \frac{d}{dx}\left[x^4 + 20x^3 - 24\right]\Big|_{x=1}$$

$$= (4x^3 + 60x^2)\Big|_{x=1} = \boxed{64}$$

$$f_y(1,2) = \frac{d}{dy}\left[f(1,y)\right]\Big|_{y=2}$$

$$= \frac{d}{dy}\left[1 + 5y^2 - 12y\right]\Big|_{y=2}$$

$$= (10y - 12)\Big|_{y=2} = \boxed{8}$$

IN GENERAL, AT ANY POINT (x,y), WE HAVE

$$f_x(x,y) = 4x^3 + 15x^2y^2$$

$$f_y(x,y) = 10x^3y - 12$$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

THERE ARE MANY NOTATIONS FOR PARTIAL DERIVATIVES. If $z = f(x,y)$ WE HAVE

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_x f = D_1 f = f_1$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = D_y f = D_2 f = f_2$$

EX. $f(x,y) = \dfrac{x-y}{x+y}$

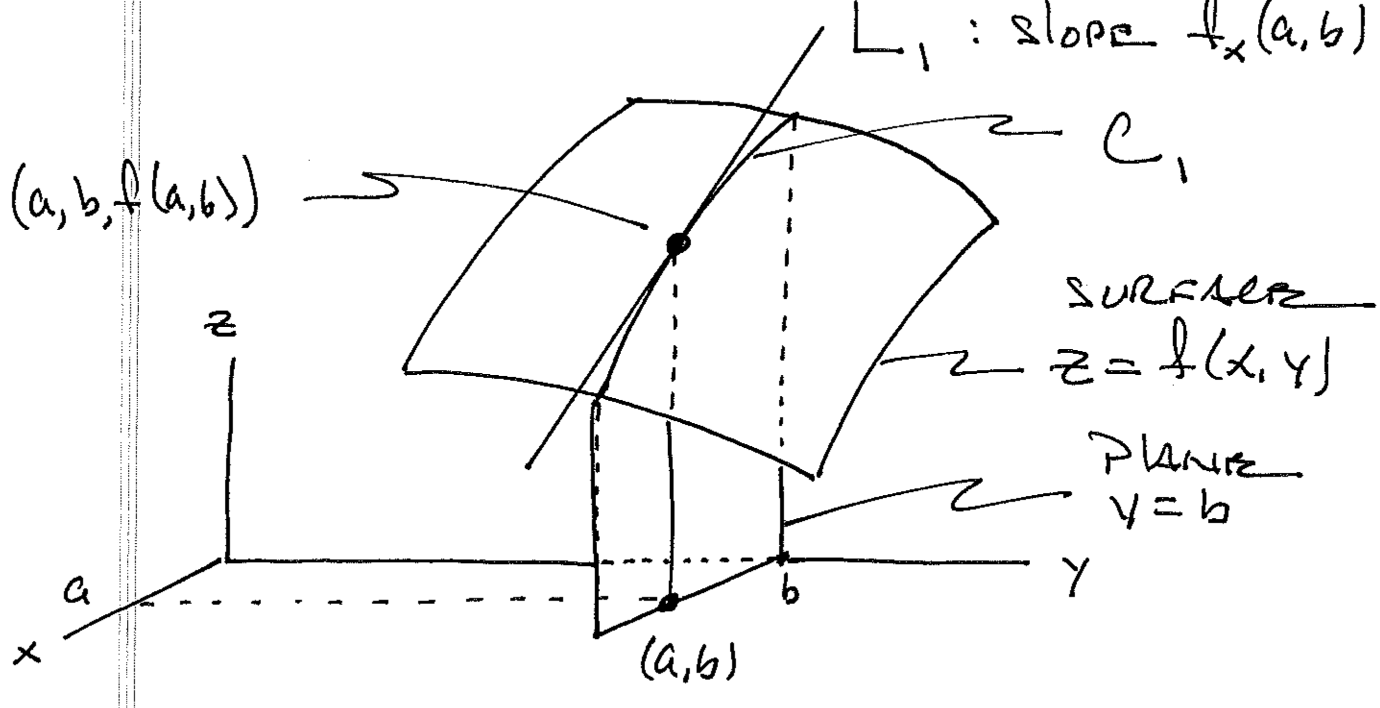
EX. $g(u,v) = \sin u \cdot \cos^2 v + u^2 v$

EX. $z = \tan(xy)$

EX. $w = (2s+3t)^{1/2}$

EX. $h(x,y,z,t) = \dfrac{xy^2}{t+3z}$

GEOMETRIC INTERPRETATION

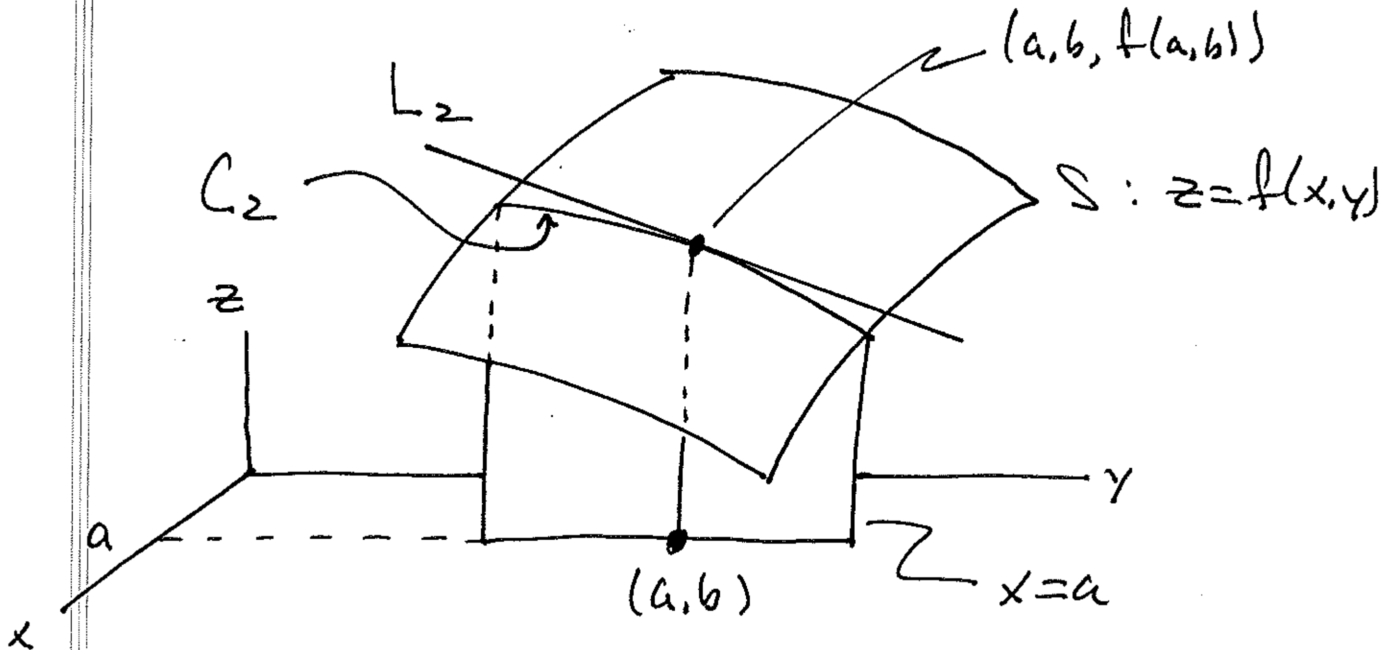


LET S BE THE SURFACE $z = f(x,y)$ AND LET C_1 BE THE CURVE OBTAINED BY INTERSECTING S WITH THE PLANE $y = b$. LET L_1 BE THE TANGENT LINE TO C_1 AT THE POINT $(a, b, f(a,b))$.

THEN THE slope OF L_1 (in THE PLANE $y = b$) is $f_x(a,b)$.

SIMILARLY, IF WE LET C_2 BE THE INTERSECTION OF S WITH THE PLANE $x = a$,

AND LET L_2 BE THE TANGENT LINE TO C_2 AT $(a, b, f(a,b))$, THEN $f_y(a,b)$ IS THE SLOPE OF L_2 IN THE PLANE $x=a$.



SIMILARLY WE DEFINE PARTIAL DERIVATIVES FOR FUNCTIONS OF 3, 4, ..., n VARIABLES

e.g. $$f_z(x,y,z) = \lim_{h \to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

e.g. $$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i+h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

if $u = f(x_1, \ldots, x_i, \ldots x_n)$.

HIGHER ORDER DERIVATIVES

THE PARTIAL DERIVATIVES OF A FUNCTION OF SEVERAL VARIABLES IS ITSELF A FUNCTION OF THE SAME NUMBER OF VARIABLES, SO WE CAN REPEAT THIS PROCESS.

NOTATION

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y\,\partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x\,\partial y}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2}$$

Ex. $f(x,y) = x^3y^5 + 3x^4y$

find $f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$

NOTICE THAT $f_{xy} = f_{yx}$.

Ex. $g(x,y) = e^{x^2 + \tan y}$

Find $g_x, g_y, g_{xx}, g_{xy}, g_{yx}, g_{yy}$

AGAIN OBSERVE $g_{xy} = g_{yx}$.

THEOREM (P. 885 CLAIRAUT)

LET $D \subseteq \mathbb{R}^2$ BE A DISK CONTAINING THE POINT (a,b), AND SUPPOSE BOTH f_{xy} AND f_{yx} ARE CONTINUOUS ON D. THEN

$$f_{xy}(a,b) = f_{yx}(a,b)$$

A GENERALIZATION OF THIS THEOREM WITH SIMILAR HYPOTHESES SAYS THAT ALL HIGHER-ORDER MIXED PARTIAL DERIVATIVES ARE INDEPENDENT OF THE ORDER IN WHICH DIFFERENTIATIONS OCCUR.

<u>EXERCISE</u>
LET $f(x,y,z) = e^{2x+3y^2+z^3}$.

SHOW (FOR INSTANCE) THAT THE FOLLOWING DERIVATIVES ARE ALL EQUAL:

$$f_{xxyz},\ f_{yxxz},\ f_{yzxx},\ f_{xyxz}$$
$$f_{xyzx},\ f_{yxzx},\ f_{xxzy},\ f_{zxxy},$$
$$f_{zyxx},\ f_{xzxy},\ f_{xzyx},\ f_{zxyx}$$

<u>EX.</u> SHOW THAT $u(x,y) = e^x \sin y$ IS A SOLUTION TO THE <u>PARTIAL DIFFERENTIAL EQUATION</u> (PDE)

$$u_{xx} + u_{yy} = 0$$

(THIS IS KNOWN AS LAPLACE'S EQUATION).

$$u_x = e^x \sin y,\quad u_{xx} = e^x \sin y$$
$$u_y = e^x \cos y,\quad u_{yy} = -e^x \sin y$$

$$\therefore\ u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0.$$

EXERCISE

SHOW THAT $v(x,y) = e^x \cos y$ IS ALSO A SOLUTION TO LAPLACE'S EQUATION, i.e.

$$v_{xx} + v_{yy} = 0.$$

EXERCISE

SHOW THAT ANY LINEAR COMBINATION OF SOLUTIONS TO LAPLACE IS AGAIN A SOLUTION. i.e. IF

$$f_{xx} + f_{yy} = 0 \quad \text{AND} \quad g_{xx} + g_{yy} = 0$$

THEN FOR ANY $a, b \in \mathbb{R}$, THE FUNCTION

$$h(x,y) = a f(x,y) + b g(x,y)$$

SATISFIES THE SAME EQUATION, i.e.

$$h_{xx} + h_{yy} = 0.$$