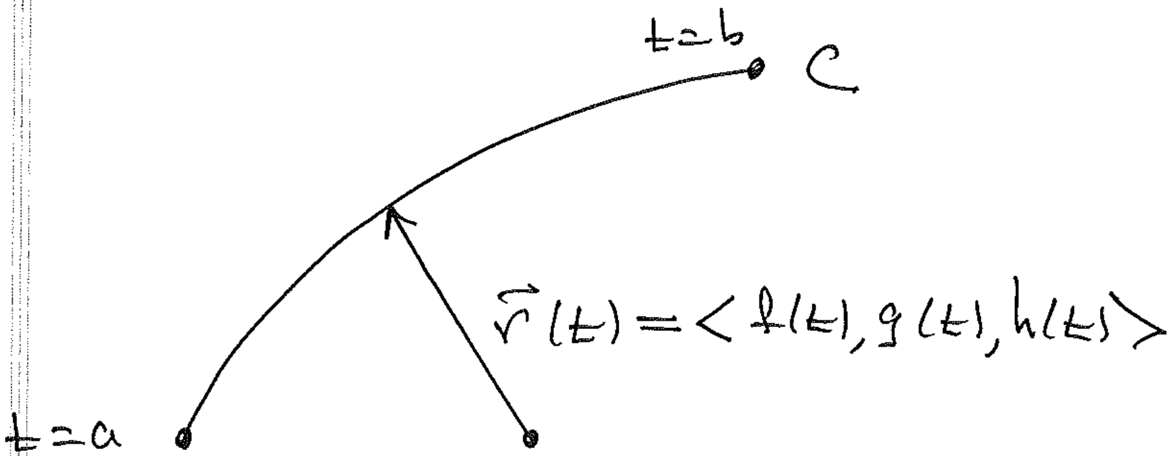
(13.3) ARC LENGTH

LET A CURVE C IN $\mathbb{R}^3$ (OR $\mathbb{R}^2$) BE PARAMETERIZED BY A VECTOR FUNCTION $\vec{r}(t)$ for $a \le t \le b$.



THM

THE LENGTH L OF C IS GIVEN BY

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

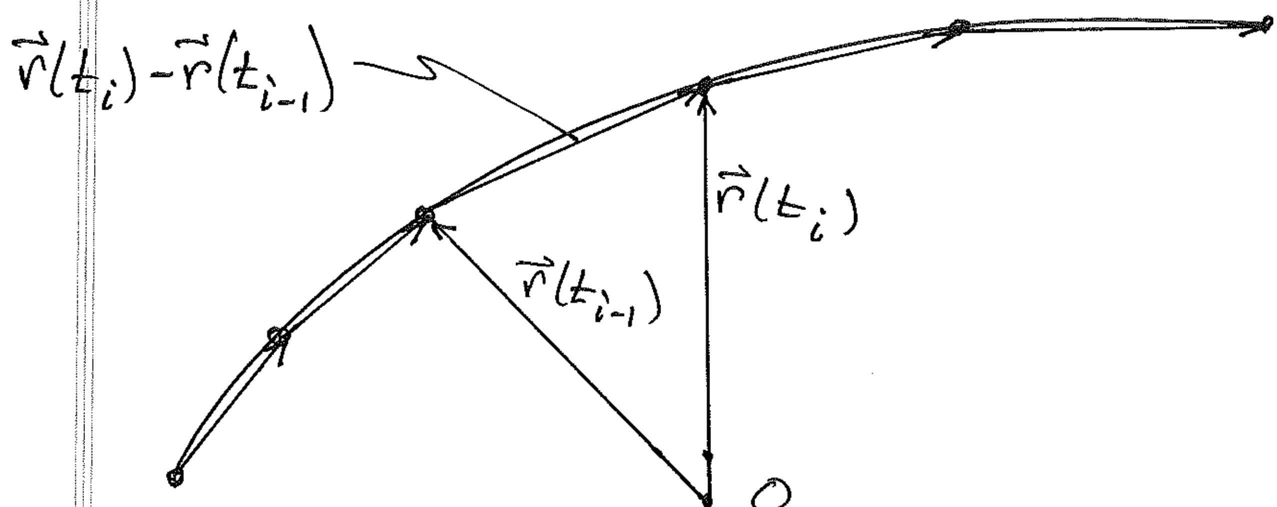
$$= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\; dt$$

TO JUSTIFY THIS FORMULA, CONSIDER A PARTITION OF $[a, b]$ INTO n SUBINTERVALS:

$$a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$$

WHERE THE i^{th} INTERVAL $[t_{i-1}, t_i]$ HAS WIDTH $\Delta t_i = t_i - t_{i-1}$ $(1 \le i \le n)$.

WE CAN USE SUCH A PARTITION TO APPROXIMATE C BY A COLLECTION OF STRAIGHT LINE SEGMENTS



THUS

$$L \approx \sum_{i=1}^{n} \left| \vec{r}(t_i) - \vec{r}(t_{i-1}) \right|$$

WE WRITE $\Delta \vec{r}_i = \vec{r}(t_i) - \vec{r}(t_{i-1})$.
THEN

$$L \approx \sum_{i=1}^{n} |\Delta \vec{r}_i| = \sum_{i=1}^{n} \left| \frac{\Delta \vec{r}_i}{\Delta t_i} \right| \cdot \Delta t_i$$

WHEN Δt_i IS SMALL WE HAVE

$$\frac{\Delta \vec{r}_i}{\Delta r_i} \approx \vec{r}\,'(t_i)$$

HENCE

$$L \approx \sum_{i=1}^{n} |\vec{r}\,'(t_i)| \, \Delta t_i$$

TAKING THE LIMIT AS $n \to \infty$ AND $\Delta t_i \to 0$ WE HAVE

$$L = \int_a^b |\vec{r}\,'(t)| \, dt$$

Ex Helix: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le 2\pi$.

$$L = \int_0^{2\pi} \sqrt{2} \, dt = 2\sqrt{2} \cdot \pi \, .$$

Ex. $\vec{r}(t) = \langle t\cos t,\ t\sin t,\ t^2 \rangle$ for $0 \le t \le 4\pi$.

RECALL THIS CURVE LIES ON THE ELLIPTIC PARABOLOID $z = x^2 + y^2$.

$$L = \int_0^{4\pi} \sqrt{5t^2+1}\, dt$$

(Ref. P. 6 # 21)

$$= \frac{1}{2} t\sqrt{5t^2+1} + \frac{1}{2\sqrt{5}} \ln\left(\sqrt{5}\, t + \sqrt{5t^2+1}\right) \Big|_0^{4\pi}$$

$$= 2\pi\sqrt{80\pi^2+1} + \frac{1}{2\sqrt{5}} \ln\left(4\sqrt{5}\,\pi + \sqrt{80\pi^2+1}\right)$$

$$= 177.57$$

Ex SET UP (ONLY) THE INTEGRAL FOR THE LENGTH OF THE CURVE C OF INTERSECTION OF

Cylinder : $x^2+y^2=3$
Plane : $2x+3y+z=2$.

Ex. $\vec{r}(t) = \langle t, t^2, t^3 \rangle \quad 1 \le t \le 2$

SET UP ONLY.

NOTE THIS CURVE IS THE INTERSECTION OF THE CYLINDERS $y = x^2$ AND $z = x^3$.

— SKIP CURVATURE —