(12.3) THE DOT PRODUCT

DEFN

GIVEN $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ THEIR DOT PRODUCT IS THE NUMBER

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

SIMILARLY, FOR VECTORS IN $\mathbb{R}^2$:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

EX. $\langle 2, 3 \rangle \cdot \langle -1, 5 \rangle = 13$

EX. $\langle -4, 0, 2 \rangle \cdot \langle 1, 12, -13 \rangle = -30$

ALGEBRAIC PROPERTIES

1.) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2.) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3.) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4.) $c(\vec{a} \cdot \vec{b}) = (c\vec{a}) \cdot \vec{b} = \vec{a} \cdot (c\vec{b})$
5.) $\vec{0} \cdot \vec{a} = 0$

EXERCISE

PROVE (1) - (5).

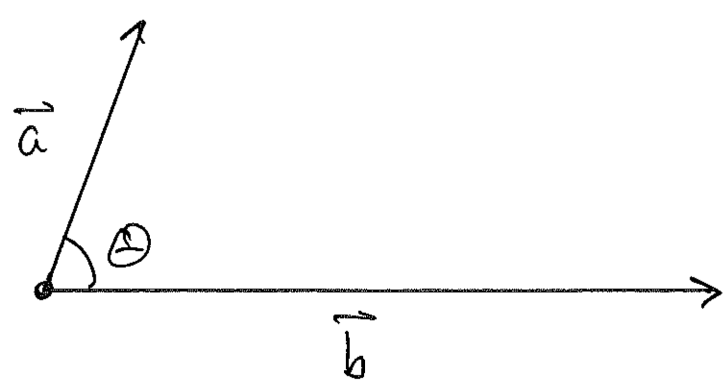
PROOF OF (3) (in $\mathbb{R}^2$)

LET $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$, AND $\vec{c} = \langle c_1, c_2 \rangle$. THEN

$$\begin{aligned}
\vec{a} \cdot (\vec{b} + \vec{c}) &= \langle a_1, a_2 \rangle \cdot \left(\langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle \right) \\
&= \langle a_1, a_2 \rangle \cdot \langle b_1 + c_1, b_2 + c_2 \rangle \\
&= a_1 (b_1 + c_1) + a_2 (b_2 + c_2) \\
&= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 \\
&= (a_1 b_1 + a_2 b_2) + (a_1 c_1 + a_2 c_2) \\
&= \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle \cdot \langle c_1, c_2 \rangle \\
&= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}
\end{aligned}$$

///

DRAW $\vec{a}$ AND $\vec{b}$ WITH THEIR ORIGIN AT A COMMON POINT, AND LET θ BE THE ANGLE FROM $\vec{a}$ TO $\vec{b}$ (IN THE PLANE SPANNED BY $\vec{a}$ and $\vec{b}$) SATISFYING $0 \leq \theta \leq \pi$.



WE CALL θ THE ANGLE BETWEEN $\vec{a}$ AND $\vec{b}$

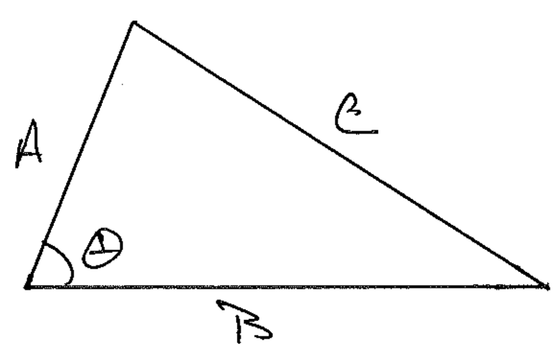
THE FOLLOWING THEOREM GIVES THE MAIN GEOMETRIC INTERPRETATION OF THE DOT PRODUCT.

THEOREM

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$$

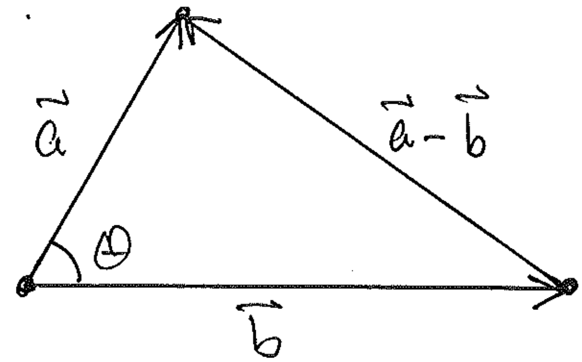
PROOF

RECALL THE LAW OF COSINES: GIVEN A TRIANGLE WITH SIDES A, B, C AND ANGLE θ OPPOSITE C:



WE HAVE: $C^2 = A^2 + B^2 - 2AB\cos\theta$.
(NOTE: THIS GENERALIZES THE THM OF PYTHAGORUS.)

APPLY THIS THM TO VECTORS $\vec{a}$, $\vec{b}$, AND $\vec{a} - \vec{b}$.



THEN

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta .$$

BUT BY THE ALGEBRAIC PROPERTIES
WE HAVE

$$\begin{aligned} |\vec{a}-\vec{b}|^2 &= (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) \\ &= \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} \\ &= |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b} \end{aligned}$$

THUS

$$|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b} = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\therefore \quad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta ,$$

AS REQUIRED. ///

TWO VECTORS $\vec{a}$, $\vec{b}$ ($\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$) ARE SAID TO BE PERPENDICULAR OR ORTHOGONAL IFF THE ANGLE BETWEEN THEM IS $\theta = \pi/2$.

WE CONSIDER $\vec{0}$ TO BE PERPENDICULAR TO ALL VECTORS. THUS

Corollary

$\vec{a}$ IS ORTHOGONAL TO $\vec{b}$ IFF

$$\vec{a} \cdot \vec{b} = 0$$

IN GENERAL WE CAN CALCULATE θ FROM THE FOLLOWING FORMULA

Corollary

GIVEN NON-ZERO VECTORS $\vec{a}$, $\vec{b}$:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

EX FIND THE ANGLE BETWEEN $\vec{a} = \langle 1, 2, -1 \rangle$ AND $\vec{b} = \langle 2, 3, 1 \rangle$.

$$\cos\theta = \frac{7}{\sqrt{6}\cdot\sqrt{14}} = \frac{\sqrt{7}}{2\cdot\sqrt{3}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{\sqrt{7}}{2\cdot\sqrt{3}}\right) = .70167 \text{ RAD}$$
$$= 40.2029 \text{ DEG.}$$

EX. FIND A VECTOR IN $\mathbb{R}^2$ WHICH IS $\perp$ TO $\vec{u} = \langle 2, 3 \rangle$.

ANS: $\langle 3, -2 \rangle$ OR $\langle -3, 2 \rangle$ OR ANY SCALAR MULTIPLE.

EX. DETERMINE A NUMBER t SUCH THAT $\vec{a} = \langle 1, -1, 7 \rangle$ IS $\perp$ TO $\vec{b} = \langle 2, 3, t \rangle$

SOLVE $\quad 2\cdot 1 + 3\cdot(-1) + t\cdot 7 = 0$

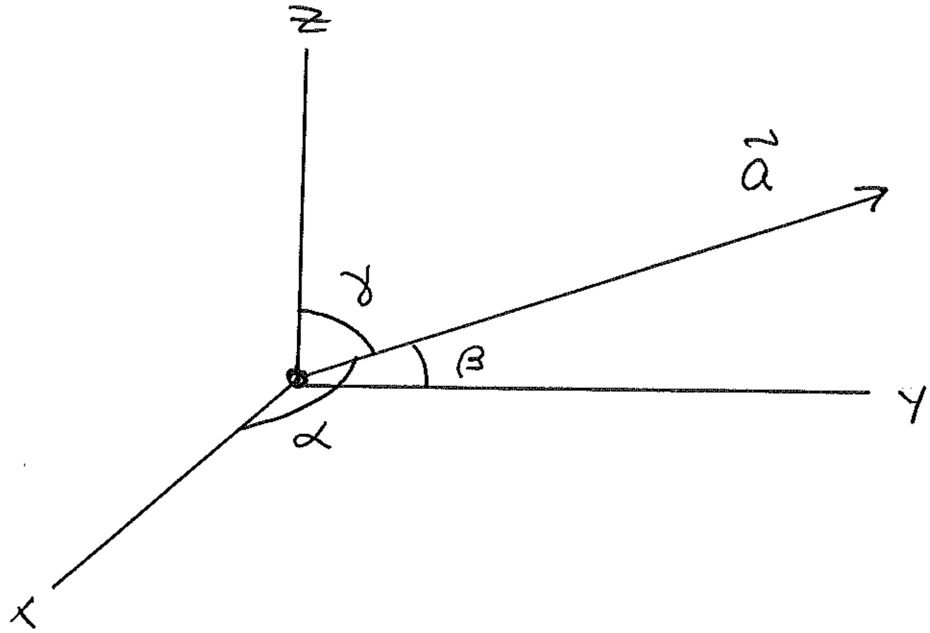
$$7t = 1 \Rightarrow \boxed{t = \frac{1}{7}}$$

DEFN

LET $\vec{a} \in \mathbb{R}^3$ BE NON-ZERO. THE DIRECTION ANGLES OF $\vec{a}$ ARE THE ANGLES α, β, γ IN $[0, \pi]$ $\vec{a}$ MAKES WITH THE POSITIVE

X, Y, AND Z - AXES, RESPECTIVELY.

THE COSINES OF α, β, γ ARE CALLED THE <u>DIRECTION COSINES</u> OF $\vec{a}$



WE HAVE, BY THE COROLLARY!

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{a_1}{|\vec{a}|}$$

$$\cos\beta = \frac{\vec{a}\cdot\vec{j}}{|\vec{a}||\vec{j}|} = \frac{a_2}{|\vec{a}|}$$

$$\cos\gamma = \frac{\vec{a}\cdot\vec{k}}{|\vec{a}||\vec{k}|} = \frac{a_3}{|\vec{a}|}$$

THUS

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$= \langle |\vec{a}| \cos\alpha, |\vec{a}| \cos\beta, |\vec{a}| \cos\gamma \rangle$$

$$= |\vec{a}| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

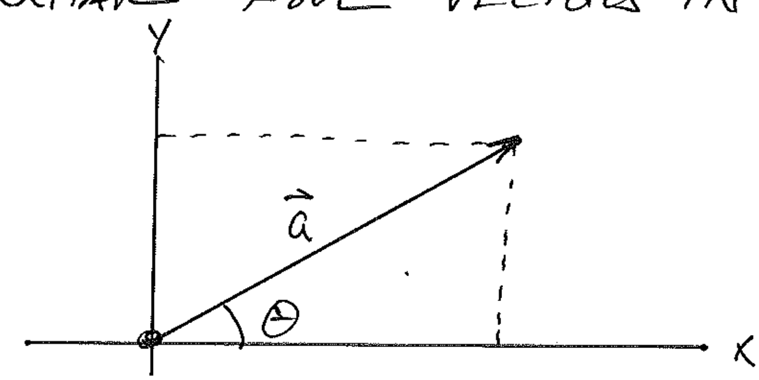
THEREFORE THE UNIT VECTOR IN THE DIRECTION $\vec{a}$ HAS COMPONENTS

$$\frac{\vec{a}}{|\vec{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle .$$

REMARK

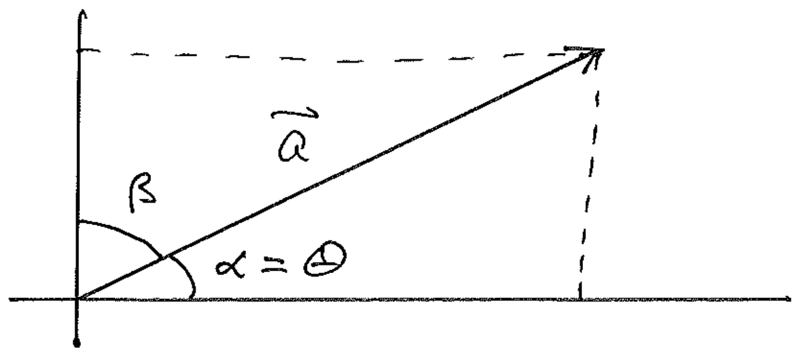
THE ABOVE DISCUSSION IS RESTRICTED TO VECTORS IN $\mathbb{R}^3$. WHY NOT $\mathbb{R}^2$? WE ALREADY DO SOMETHING VERY SIMILAR FOR VECTORS IN $\mathbb{R}^2$.

RECALL :



$$\vec{a} = \langle |\vec{a}| \cos\theta, |\vec{a}| \sin\theta \rangle$$

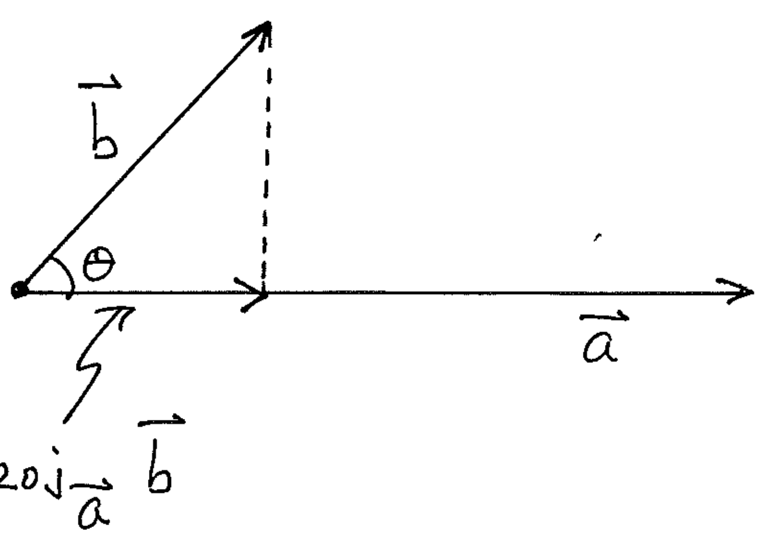
LET $\alpha = \theta$, AND LET β BE THE ANGLE $\vec{a}$ MAKES WITH THE POSITIVE y-AXIS THEN $\alpha + \beta = \pi/2$, SO THAT $\sin\theta = \sin\alpha = \cos\beta$



$\therefore \quad \vec{a} = \langle |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta \rangle$

DEFN

LET $\vec{a}$ AND $\vec{b}$ BE NON-ZERO VECTORS. THE PROJECTION OF $\vec{b}$ ON $\vec{a}$ IS THE VECTOR PARALLEL TO $\vec{a}$ WHICH FORMS THE BASE OF RIGHT TRIANGLE HAVING HYPOTENUSE $\vec{b}$.



ITS LENGTH IS CALLED THE <u>COMPONENT OF $\vec{b}$ ALONG $\vec{a}$</u>.

$$\text{Comp}_{\vec{a}}\,\vec{b} = \left|\,\text{Proj}_{\vec{a}}\,\vec{b}\,\right|$$

OBSERVE FROM THE FIGURE THAT

$$\text{Comp}_{\vec{a}}\,\vec{b} = |\vec{b}|\cos\theta = \cancel{|\vec{b}|}\cdot\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\cancel{|\vec{b}|}}$$

$$\therefore \quad \boxed{\text{Comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

HENCE

$$\text{Proj}_{\vec{a}}\,\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\cdot\frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

$$\therefore \quad \boxed{\text{Proj}_{\vec{a}}\,\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}}$$

<u>Ex.</u> $\vec{a} = \langle 2, 3, -1 \rangle$
$\vec{b} = \langle 5, 1, -2 \rangle$

$$\text{Comp}_{\vec{a}} \vec{b} = \frac{10+3+2}{\sqrt{4+9+1}} = \frac{15}{\sqrt{14}}$$

$$\text{Proj}_{\vec{a}} \vec{b} = \left(\frac{15}{14}\right)\langle 2, 3, -1 \rangle = \left\langle \frac{15}{7}, \frac{45}{14}, -\frac{15}{14} \right\rangle.$$

<u>Ex.</u> LET $\vec{a} = \langle 1, 1 \rangle$, $\vec{b} = \langle 2, -1 \rangle$.
DETERMINE A VECTOR $\vec{c}$ SUCH THAT

$$\text{comp}_{\vec{a}} \vec{c} = \text{comp}_{\vec{b}} \vec{c} = 1 .$$

<u>SOLUTION</u>: LET $\vec{c} = \langle s, t \rangle$. THEN

$$\frac{\vec{a} \cdot \vec{c}}{|\vec{a}|} = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}|} = 1$$

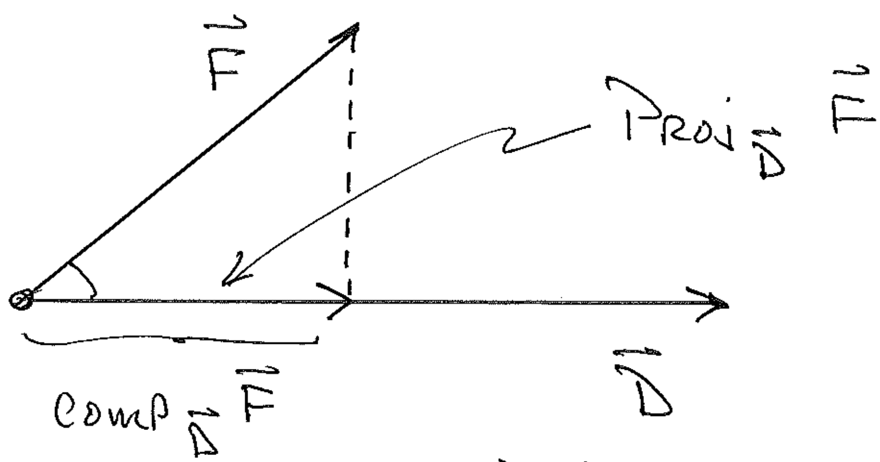
$$\Rightarrow \begin{cases} \vec{a} \cdot \vec{c} = |\vec{a}| \\ \vec{b} \cdot \vec{c} = |\vec{b}| \end{cases}$$

$$\Rightarrow \begin{cases} s + t = \sqrt{2} \\ 2s - t = \sqrt{5} \end{cases} \Rightarrow s = \frac{\sqrt{2}+\sqrt{5}}{3} \;\; ; \;\; t = \frac{2\sqrt{2}-\sqrt{5}}{3}$$

$$\therefore \quad \vec{c} = \left\langle \frac{\sqrt{2}+\sqrt{5}}{3}, \frac{2\sqrt{2}-\sqrt{5}}{3} \right\rangle$$

SUPPOSE A FORCE $\vec{F}$ ACTS ON AN OBJECT WHILE THAT OBJECT MOVES ALONG A DISPLACEMENT VECTOR $\vec{D}$. THE <u>WORK</u> DONE BY $\vec{F}$ DURING THIS MOTION IS DEFINED TO BE

$$\boxed{W = \left(\text{comp}_{\vec{D}}\vec{F}\right)\cdot|\vec{D}|}$$

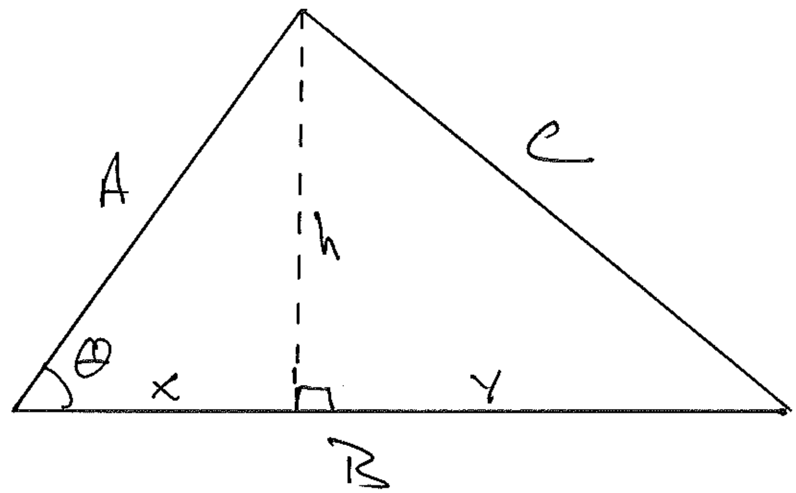


OBSERVE $W = \dfrac{\vec{F}\cdot\vec{D}}{|\vec{D}|}\cdot|\vec{D}| = \vec{F}\cdot\vec{D}$.

i.e. $\boxed{W = \vec{F}\cdot\vec{D}}$

PROOF OF LAW OF COSINES

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$



PROOF:

FROM THE FIGURE WE HAVE

$$\begin{cases} C^2 = h^2 + y^2, & A^2 = h^2 + x^2 \\ B = x + y, & A\cos\theta = x \end{cases}$$

THUS

$$A^2 + B^2 - 2AB\cos\theta$$

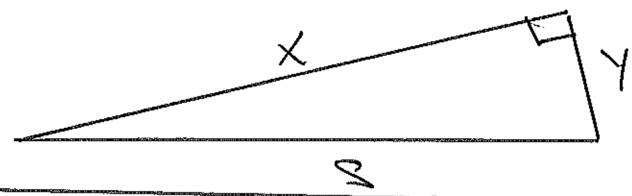
$$= (h^2 + x^2) + (x+y)^2 - 2(x+y)x$$

$$= h^2 + x^2 + x^2 + 2xy + y^2 - 2x^2 - 2xy$$

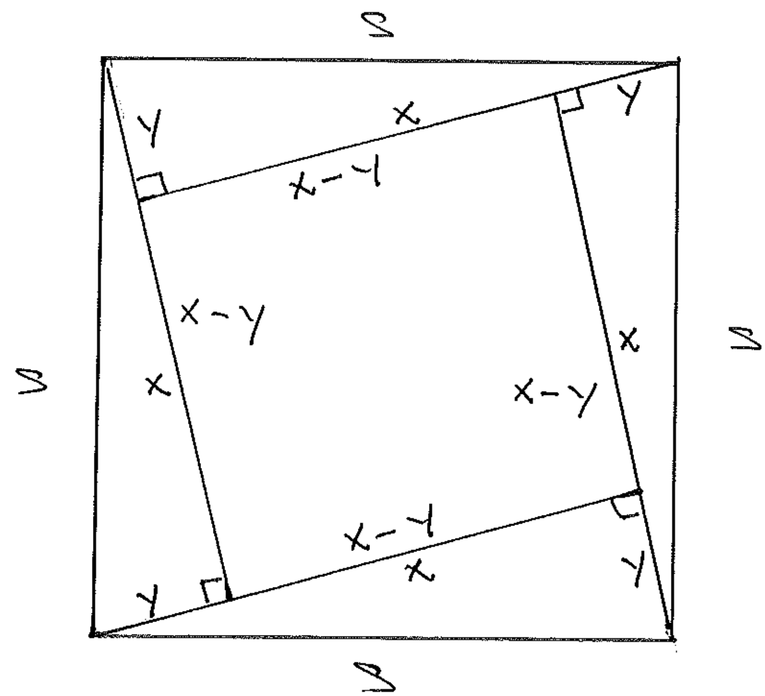
$$= h^2 + y^2$$

$$= C^2 \qquad ///$$

PROOF OF PYTHAGOREAN THEOREM



show $\boxed{s^2 = x^2 + y^2}$.



PROOF:

$$\begin{aligned} s^2 &= \text{area of large square} \\ &= 4\cdot(\text{area of triangle}) + (\text{area of small sq.}) \\ &= 4\cdot\left(\tfrac{1}{2}xy\right) + (x-y)^2 \\ &= \cancel{2xy} + x^2 - \cancel{2xy} + y^2 \\ &= x^2 + y^2. \end{aligned}$$

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