MATH 21
Linear Algebra
Winter 2007

Midterm 2 Review Problems

1. Determine whether the following sets are linearly independent or linearly dependent. If linearly dependent, write a non trivial relation involving the vectors in the set.

   a. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix} \right\}$

   b. $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right\}$

   c. $\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 10 \\ 2 \\ 0 \\ 12 \\ 3 \end{pmatrix} \right\}$

2. Determine the rank and nullity of each of the following matrices.

   a. $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 7 \\ 4 & 8 & 12 \end{pmatrix}$

   b. $\begin{pmatrix} 0 & 1 & 1 & 6 \\ 1 & 0 & 2 & 7 \\ 2 & 2 & 5 & -2 \end{pmatrix}$

   c. $\begin{pmatrix} 2 & 5 & -1 \\ 1 & 2 & -2 \\ 7 & 4 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

3. Determine a basis for (i) the image, and (ii) the kernel of each of the matrices in the previous problem.

4. Let V be the set of vectors in $\mathbf{R}^3$ which are perpendicular to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   a. Show directly that V is a subspace of $\mathbf{R}^3$.
   b. Determine a basis for V. (Hint: observe that $V = \ker(1 \quad 1 \quad 1)$, and proceed as in problem (3).)

5. Let $\vec{x}, \vec{y} \in \mathbf{R}^n$. Answer the following two questions

I. Does there exist an *invertible* linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ such that $T(\vec{x}) = \vec{y}$,
II. Does there exist a *non-invertible* linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ such that $T(\vec{x}) = \vec{y}$,

in the following four cases:

a. $\vec{x} \neq \vec{0}$ and $\vec{y} \neq \vec{0}$
b. $\vec{x} \neq \vec{0}$ and $\vec{y} = \vec{0}$
c. $\vec{x} = \vec{0}$ and $\vec{y} \neq \vec{0}$
d. $\vec{x} = \vec{0}$ and $\vec{y} = \vec{0}$

(i.e. eight questions altogether). If an answer is yes, give an example establishing the existence of such a linear map. If no, explain why no linear map with the given property can exist.

6. Let $T : \mathbf{R}^m \to \mathbf{R}^n$ be a linear transformation. Prove that T is injective if and only if $\ker(T) = \{\vec{\mathbf{0}}\}$.

7. Let $T : \mathbf{R}^m \to \mathbf{R}^n$ be a linear transformation. Prove that
a. $\mathrm{im}(T)$ is a subspace of $\mathbf{R}^n$
b. $\ker(T)$ is a subspace of $\mathbf{R}^m$

8. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$, and $\vec{x} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$.
a. Show that $\mathcal{B}$ is a basis for $\mathbf{R}^3$.
b. Determine $[\vec{x}]_{\mathcal{B}}$, the coordinate vector of $\vec{x}$ with respect to $\mathcal{B}$.

9. Let $T(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 4 & 1 \\ 3 & 7 \end{pmatrix}$. Determine the matrix B of T with respect to the basis $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ of $\mathbf{R}^2$. Verify that $B[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}$ for all $\vec{x} \in \mathbf{R}^2$.

10. Let $\mathcal{B} = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \}$ be a basis for a subspace V of $\mathbf{R}^n$. Show that $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$ for all $\vec{x}, \vec{y} \in \mathbf{R}^n$.

11. Let A and B be two square matrices, and suppose that A is similar to B. Prove the following.
a. If t is a non-negative integer, then A^t is similar to B^t
b. $\mathrm{rank}(A) = \mathrm{rank}(B)$. (Hint: let $B = S^{-1}AS$, $r = \mathrm{rank}(B)$, and let $\{\vec{u}_1, \ldots, \vec{u}_r\}$ be a basis for $\mathrm{im}(B)$. Show that $\{S\vec{u}_1, \ldots, S\vec{u}_r\}$ is a basis for $\mathrm{im}(A)$, whence $\mathrm{rank}(A) = r$ also. Alternate hint: let $p =$ nullity(B), and suppose $\{\vec{w}_1, \ldots, \vec{w}_p\}$ is a basis for $\ker(B)$. Show that $\{S\vec{w}_1, \ldots, S\vec{w}_p\}$ is a basis for $\ker(A)$, whence nullity(A)$= p$ also, and hence $\mathrm{rank}(A) = \mathrm{rank}(B)$.)

12. Fix an $n \times n$ matrix A, and let $V = \{B \in M_n \mid AB = BA\}$, i.e. V is the set of all matrices which commute with A. Show that V is a subspace of M_n.

13. Let $V = \{f \in F(\mathbf{R}, \mathbf{R}) \mid f(6) = 0\}$. Show that V is a subspace of $F(\mathbf{R}, \mathbf{R})$.

14. Fix $A \in M_n$, and define $T : M_n \to M_n$ by $T(B) = AB$. Show
    a. T is a linear transformation
    b. $\ker(T) = \{B \in M_n \mid \text{im}(B) \subseteq \ker(A)\}$
    c. T is invertible if and only if A is invertible