MATH 21
Linear Algebra
Winter 2007

Midterm 1 Review Problems

1. Determine all solutions to the following linear systems of equations.
   a. $\begin{cases} 4x + 3y = 2 \\ 7x + 5y = 3 \end{cases}$
   b. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 3 & 4 & 3 \\ 1 & 4 & 5 & 4 \end{array}\right)$
   c. $\begin{cases} x_1 + 2x_2 \quad + 2x_4 + 3x_5 = 0 \\ \qquad x_3 + 3x_4 + 2x_5 = 0 \\ \qquad x_3 + 4x_4 - x_5 = 0 \\ \qquad\qquad x_5 = 0 \end{cases}$

2. Find the reduced row echelon form (RREF) of each of the following matrices.
   a. $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 7 \\ 4 & 8 & 12 \end{pmatrix}$
   b. $\begin{pmatrix} 0 & 1 & 1 & 6 \\ 1 & 0 & 2 & 7 \\ 2 & 2 & 5 & -2 \end{pmatrix}$

3. Determine all possible RREFs of a 3×3 matrix. Show the possible positions of all leading 1s, all 0s, and write $*$ for an arbitrary entry. For instance one such RREF having rank 2 would be

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix}$$

4. Let A be an $n \times m$ matrix, and let $\mathrm{r} = \mathrm{rank}(A)$.
   a. Explain why necessarily $r \le n$ and $r \le m$.
   b. Explain why the system $\left(A \mid \vec{b}\right)$ is inconsistent if and only if $\mathrm{RREF}(A)$ contains the row $\left(0 \cdots 0 \mid 1\right)$.
   c. Suppose $r < m$. Explain why the system $\left(A \mid \vec{b}\right)$ cannot have a unique solution.
   d. Suppose $r = m$. Explain why the system $\left(A \mid \vec{b}\right)$ must have at least one solution.

5. A linear system $\left(A \mid \vec{b}\right)$ has 60 rows and 91 columns, and we are told that $\mathrm{rank}(A) = 28$. How many free variables will the solution have?

6. Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in \mathbf{R}$. Show that $(A\vec{u}) \bullet (A\vec{v}) = \vec{u} \bullet \vec{v}$ for any $\vec{u},\ \vec{v} \in \mathbf{R}^2$.

7. Let X, Y, and Z be sets, and $f : X \to Y$, and $g : Y \to Z$ functions. Show that if $g \circ f$ is injective, then f is also injective.

8. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Show directly that $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$.

9. Let $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find a unit vector $\vec{u}$ in the direction $\vec{v}$. Let L be the line through the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ with direction $\vec{v}$. Write the matrix for the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by orthogonal projection onto L. Is this transformation invertible? If so give its inverse, if not explain why not.

10. Determine all real numbers α such that the follwing are true:
    a. $\begin{pmatrix} 2 & 3 \\ 5 & \alpha \end{pmatrix}$ is invertible.
    b. $\begin{pmatrix} \alpha & 3 \\ 12 & \alpha \end{pmatrix}$ is not invertible.
    c. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & \alpha \\ 1 & 4 & \alpha^2 \end{pmatrix}$ is not invertible.

11. Find the inverses of the following matrices, if they exist, or explain why they don't exist.
    a. $\begin{pmatrix} 3 & 1 \\ 7 & -1 \end{pmatrix}$
    b. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$
    c. $\begin{pmatrix} 2 & 1 & 5 & -12 \\ 14 & 11 & 3 & -1 \\ 6 & 0 & -6 & 5 \end{pmatrix}$

12. Let A, and B be $n \times n$ matrices, let O denote the $n \times n$ zero matrix (i.e. the $n \times n$ matrix all of whose entries are zero), and let M be the $(2n) \times (2n)$ partitioned matrix

$$M = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

Prove that M is invertible if and only if both A and B are invertible, and write a formula for M^{-1} in terms of A^{-1} and B^{-1}.

13. Compute the following matrix products.

a. $\begin{pmatrix} 1 & -2 & -5 \\ -2 & 5 & 11 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 & 4 \\ 0 & -1 & 5 & 0 \\ 6 & 4 & 2 & 1 \end{pmatrix}$

b. $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

c. $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

14. Find a 2×2 matrix A such that $A \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -10 \end{pmatrix}$.

15. Let A be an $n \times p$ matrix, B be a $p \times m$ matrix, and let $\alpha \in \mathbf{R}$. Prove that $\alpha(AB) = (\alpha A)B$.