MATH 21
Linear Algebra
Winter 2007

Final Review Problems

These problems concentrate on the material covered since the second midterm. Please refer to the earlier review sheets for earlier material. The Final exam will be cumulative, with a slight emphasis on the most recent material. Expect between 10 and 12 problems with a wide range of difficulties: some purely computational, some proofs.

1. Let V be a linear space and let $\{f_1, \ldots, f_n\} \subseteq V$.
   a. Define $\operatorname{span}(f_1, \ldots, f_n)$.
   b. Define what it means for $W \subseteq V$ to be a subspace of V.
   c. Show that $\operatorname{span}(f_1, \ldots, f_n)$ is a subspace of V.
   d. Define what it means for $\{f_1, \ldots, f_n\}$ to be linearly independent
   e. Define what it means for $\{f_1, \ldots, f_n\}$ to span V.
   f. Define what it means for $\{f_1, \ldots, f_n\}$ to be a basis for V.
   g. Suppose $\mathcal{B} = \{f_1, \ldots, f_n\}$ is a basis. Define the $\mathcal{B}$-coordinate transformation from V to $\mathbf{R}^n$.
   h. Again supposing $\mathcal{B} = \{f_1, \ldots, f_n\}$ is a basis, show that the $\mathcal{B}$-coordinate transformation is a linear isomorphism.
   i. Let $\mathcal{A} = \{g_1, \ldots, g_n\}$ and $\mathcal{B} = \{f_1, \ldots, f_n\}$ be two bases of V. Define the change of basis matrix S which transforms $\mathcal{B}$-coordinates into $\mathcal{A}$-coordinates. State and prove a formula which gives the columns of S in terms of the $\mathcal{A}$-coordinates of $\mathcal{B}$.
   j. Let $T : V \to V$ be a linear transformation and $\mathcal{B} = \{f_1, \ldots, f_n\}$ a basis. Define the matrix B of T with respect to $\mathcal{B}$. State and prove a formula giving the columns of B in terms of the $\mathcal{B}$-coordinate transformation.
   k. Let $T : V \to V$ be a linear transformation, let $\mathcal{A} = \{g_1, \ldots, g_n\}$ and $\mathcal{B} = \{f_1, \ldots, f_n\}$ be two bases for V, and let A and B be the matrices of T with respect to $\mathcal{A}$ and $\mathcal{B}$, respectively. Let S be the change of basis matrix from $\mathcal{B}$-coordinates into $\mathcal{A}$-coordinates. Write a formula that gives the relationship between matrices A, B, and S. Draw a commutative diagram which justifies this formula.

2. Let V and W be linear spaces, and let $T : V \to W$ be a linear transformation. ().
   a. Let T be injective, and $\{f_1, \ldots, f_n\} \subseteq V$ linearly independent. Prove that $\{T(f_1), \ldots, T(f_n)\} \subseteq W$ is also linearly independent.
   b. Let T be surjective, and suppose $\{f_1, \ldots, f_n\}$ spans V. Prove that $\{T(f_1), \ldots, T(f_n)\}$ spans W.
   c. Let T be an isomorphism, and suppose $\{f_1, \ldots, f_n\}$ is a basis of V. Prove that $\{T(f_1), \ldots, T(f_n)\}$ is a basis of W.
   d. Suppose V is finite dimensional, and that T is an isomorphism. Prove that W is also finite dimensional, and that $\dim(V) = \dim(W)$.
   e. Prove that T is an isomorphism if and only if both $\ker(T) = \{0\}$ and $\operatorname{im}(T) = W$.

   Hint: see pages 149-150 of the lecture notes.

3. Let U, V, and W be linear spaces. We write $U \approx V$ to mean that U is isomorphic to V. Prove that the isomorphism relation possesses the following properties.
   a. Reflexive: $U \approx U$
   b. Symmetric: $U \approx V$ implies $V \approx U$
   c. Transitive: $U \approx V$ and $V \approx W$ implies $U \approx W$

4. Let $P_3 = \{\text{polynomials of degree} \le 3\}$, and let $D: P_3 \to P_3$ denote differentiation (defined by $D(x^n) = nx^{n-1}$ for any $n \ge 0$.)
   a. Write the matrix B of D with respect to the basis $\{1, x, x^2, x^3\}$ of P_3.
   b. Use the matrix B found in (a) to determine the rank and nullity of D.
   c. Write bases for the subspaces $\operatorname{im}(D)$ and $\ker(D)$ of P_3.
   d. Show that $\{1 - x - x^3, x - x^2, x^2, x^3\}$ is also a basis of P_3.
   e. Write the matrix A of D with respect to $\{1 - x - x^3, x - x^2, x^2, x^3\}$.
   f. Use the matrix A found in (e) to determine the rank and nullity of D, confirming the answer to (b).
   g. Write the change of basis matrix S from $\{1, x, x^2, x^3\}$ to $\{1 - x - x^3, x - x^2, x^2, x^3\}$.
   h. Verify that $SB = AS$.

5. Let M_2={all 2×2 matrices}, and let $\mathcal{B} = \{E_1, E_2, E_3, E_4\}$ be the standard basis for M_2, given by $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
   a. Determine a matrix $B \in M_2$ such that the linear transformation $T: M_2 \to M_2$ defined by $T(A) = AB - BA$ for all $A \in M_2$ has matrix $\begin{pmatrix} 0 & 2 & -2 & 0 \\ 2 & 2 & 0 & -2 \\ -2 & 0 & -2 & 2 \\ 0 & -2 & 2 & 0 \end{pmatrix}$ with respect to $\mathcal{B}$.
   b. Show that $\mathcal{C} = \{F_1, F_2, F_3, F_4\}$, where $F_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $F_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $F_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, is also a basis for M_2.
   c. Find the change of basis matrices from $\mathcal{C}$ to $\mathcal{B}$, and from $\mathcal{B}$ to $\mathcal{C}$.
   d. Find the matrix of T with respect to $\mathcal{C}$.

6. Find the determinants of the following matrices.

   a. $\begin{pmatrix} -2 & 1 & -1 \\ -3 & 1 & 4 \\ 5 & 2 & 6 \end{pmatrix}$
   b. $\begin{pmatrix} 3 & -4 & 2 & 1 \\ 2 & -2 & 1 & 1 \\ 1 & 3 & 2 & -1 \\ -1 & 1 & 2 & 5 \end{pmatrix}$
   c. $\begin{pmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$

   d. $\begin{pmatrix} 2 & -23 & 6 & 17 & 11 \\ 0 & 4 & 3 & 12 & 12 \\ 0 & 0 & -1 & -50 & 43 \\ 0 & 0 & 0 & 4 & 6 \\ 3 & 0 & 0 & 0 & 1 \end{pmatrix}$
   e. $\begin{pmatrix} -2 & 1 & -1 & 0 & 0 & 0 \\ -3 & 1 & 4 & 0 & 0 & 0 \\ 5 & 2 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -12 & -13 & 3 \end{pmatrix}$

7. In the following, use the fact that $\det(AB) = \det(A)\det(B)$, for any square matrices A and B.

   a. Prove that if S is an invertible square matrix, then $\det(S^{-1}) = \dfrac{1}{\det(S)}$.

   b. Prove that if square matrices A and B are similar, then $\det(A) = \det(B)$.

   c. To define the determinant of a linear transformation $T : V \to V$, pick a basis $\mathcal{B} = \{f_1, \ldots, f_n\}$ of V, then form the matrix B of T with respect to $\mathcal{B}$. We then define $\det(T)$ to be $\det(B)$. Explain why this definition is independent of the choice of basis $\mathcal{B}$, i.e. explain why we would get the same number for $\det(T)$ if we had chosen a different basis.

   d. Find the determinants of the linear transformations in problems (4) and (5) of this review sheet.

   e. Find the determinant of the linear transformation $T : M_2 \to M_2$ given by $T(A) = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} A$ for all $A \in M_2$.

8. Find all real eigenvalues of the following matrices, along with their algebraic multiplicities.

   a. $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$ b. $\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$ c. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ d. $\begin{pmatrix} 5 & 1 & -5 \\ 2 & 1 & 0 \\ 8 & 2 & -7 \end{pmatrix}$

9. Find a basis for the eigenspace belonging to each of the real eigenvalues found in problem (8). Determine the geometric multiplicity of each eigenvalue.

10. Let $A = \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix}$. Determine the eigenvalues and associated eigenvectors of A. Determine a basis of $\mathbf{R}^2$ with respect to which, A becomes diagonal. Write the matrix of A with respect to that basis.