7.2 FINDING EIGENVALUES OF A MATRIX

LET $A \in M_n$. OBSERVE

$$
\begin{aligned}
& A\vec{v} = \lambda \vec{v} \quad \text{FOR SOME NON-ZERO } \vec{v} \in \mathbb{R}^n \\
\iff & A\vec{v} = \lambda I_n \vec{v} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \\
\iff & A\vec{v} - \lambda I_n \vec{v} = \vec{0} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \\
\iff & (A - \lambda I_n)\vec{v} = \vec{0} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \\
\iff & \ker(A - \lambda I_n) \neq \{\vec{0}\} \\
\iff & \det(A - \lambda I_n) = 0
\end{aligned}
$$

THEOREM

$\lambda \in \mathbb{R}$ IS AN EIGENVALUE OF A IF AND ONLY IF

$$\det(A - \lambda I_n) = 0.$$

THIS IS CALLED THE CHARACTERISTIC EQUATION FOR A. IT'S SOLUTIONS λ ARE PRECISELY THE EIGENVALUES OF A.

EX. $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

EX. $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{pmatrix}$

THEOREM

THE EIGENVALUES OF AN UPPER (OR LOWER) TRIANGULAR MATRIX ARE ITS DIAGONAL ENTRIES.

PROOF: EXERCISE.

EX.

$$A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{pmatrix}$$

CHARACTERISTIC POLYNOMIAL $\longrightarrow$ $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

$\lambda = 1$: EIGENVALUE OF (ALGEBRAIC) MULTIPLICITY 1
$\lambda = 2$: " " " 2.

EX. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det(A - \lambda I_2) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$ad - (a+d)\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

DEFN: THE TRACE OF $A \in M_n$ IS THE SUM OF ITS DIAGONAL ENTRIES. IF

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

THEN

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

THUS THE CHARACTERISTIC EQN. OF A 2x2 MATRIX IS

$$det(A - \lambda I_2) = \lambda^2 - tr(A)\lambda + det(A) = 0$$

EX $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ $tr(A) = 4$, $det(A) = -5$

CHAR EQN: $\lambda^2 - 4\lambda - 5 = 0$

THEOREM

IF $A \in M_n$ THEN $det(A - \lambda I_n)$ IS A POLYNOMIAL OF DEGREE n (IN λ) OF THE FORM

$$f_A(\lambda) = (-\lambda)^n + tr(A)(-\lambda)^{n-1} + \cdots + det(A)$$

WE CALL $f_A(\lambda)$ THE CHARACTERISTIC POLYNOMIAL FOR A.

Ex.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\det(A - \lambda I_3) = (1-\lambda)(\lambda^2 + \lambda + 1) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \quad \lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

Hw (7.2) 2-12 even, 16, 18