4.1 LINEAR SPACES

A <u>LINEAR SPACE</u> (ALSO <u>VECTOR SPACE</u>) IS A SET V TOGETHER WITH TWO BINARY OPERATIONS

$$+ : V \times V \rightarrow V \quad \text{(ADDITION)}$$
$$\cdot : \mathbb{R} \times V \rightarrow V \quad \text{(SCALAR MULTIPLICATION)}$$

SATISFYING THE FOLLOWING AXIOMS FOR ALL $f, g, h \in V$ AND $k, c \in \mathbb{R}$.

1.) ASSOCIATIVE $(+)$: $(f+g)+h = f+(g+h)$.
2.) COMMUTATIVE $(+)$: $f+g = g+h$.
3.) ZERO ELEMENT: THERE EXISTS AN ELEMENT $0 \in V$ SUCH THAT $f+0=f$.
4.) INVERSE ELEMENTS: FOR EACH $f \in V$ THERE EXISTS AN ELEMENT $(-f) \in V$ SUCH THAT $f+(-f)=0$.
5.) DISTRIBUTIVE $(\cdot$ OVER $+)$: $k(f+g) = kf+kg$.
6.) DISTRIBUTIVE $(\cdot$ OVER $+(\mathbb{R}))$: $(c+k)f = cf+kf$.
7.) ASSOCIATIVE $(\cdot$ WITH $\cdot(\mathbb{R}))$: $c(kf) = (ck)f$.
8.) IDENTITY $(\cdot)$: $1 \cdot f = f$.

THIS DEFINITION REPRESENTS A REAL STEP UP IN ABSTRACTION, SINCE THE SYMBOLS $+$ AND $\cdot$ ARE NOW TO BE REGARDED AS VARIABLES.

EX. $V = \mathbb{R}^n$

$+$ ADDITION OF VECTORS

$\cdot$

0 ZERO VECTOR $\vec{0}$

IF $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ THEN $-\vec{x} = \begin{pmatrix} -x_1 \\ \vdots \\ -x_n \end{pmatrix}$

AXIOMS 1-8 ARE OBVIOUSLY SATISFIED.

EX. LET $F(\mathbb{R},\mathbb{R})$ DENOTE THE SET OF ALL FUNCTIONS $f:\mathbb{R}\to\mathbb{R}$. DEFINE

$$(f+g)(x) = f(x)+g(x)$$

AND

$$(k\cdot f)(x) = k f(x)$$

FOR ANY $f,g \in F(\mathbb{R},\mathbb{R})$, $k\in\mathbb{R}$ AND $x\in\mathbb{R}$. THE ZERO FUNCTION IS SIMPLY

$$0(x) = 0 \quad \text{FOR ALL } x\in\mathbb{R}$$

AND FOR ANY $f\in F(\mathbb{R},\mathbb{R})$ DEFINE $-f$ BY

$$(-f)(x) = -f(x).$$

WITH THESE DEFINITIONS $F(\mathbb{R},\mathbb{R})$ IS A LINEAR SPACE.

Ex. LET M_{nm} DENOTE THE SET OF ALL $n \times m$ MATRICES. WRITE

$$A = (a_{ij}) \quad \text{AND} \quad B = (b_{ij})$$

WHERE $1 \le i \le n$, $1 \le j \le m$. DEFINE

$$A + B = (a_{ij} + b_{ij})$$
$$kA = (ka_{ij})$$
$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix}$$

AND

$$-A = (-a_{ij})$$

AXIOMS 1–8 ARE READILY VERIFIED SO M_{nm} IS A LINEAR SPACE.

Ex. LET $\mathcal{L}$ BE THE SET OF ALL INFINITE SEQUENCES OF REAL NUMBERS

$$(x_1, x_2, x_3, \ldots) = (x_i)_{i=1}^{\infty}$$

DEFINE

$$(x_i) + (y_i) = (x_i + y_i)$$
$$k \cdot (x_i) = (kx_i)$$
$$O = (0)_{i=1}^{\infty} = (0, 0, \ldots)$$
$$-(x_i) = (-x_i)$$

THEN $\mathcal{L}$ is clearly A LINEAR SPACE.

NOTE THAT $\mathcal{L}$ IS IDENTICAL TO $F(\mathbb{Z}^+, \mathbb{R})$, THE SET OF ALL FUNCTIONS $f: \mathbb{Z}^+ \to \mathbb{R}$. REGARD f TO BE A SEQUENCE WHOSE i^{TH} TERM IS $f(i) = f_i$.

$$f = (f_i)_{i=1}^{\infty} = (f_1, f_2, f_3, \ldots\ldots)$$

CONVERSELY, ANY SUCH SEQUENCE GIVES RISE TO A FUNCTION $f: \mathbb{Z}^+ \to \mathbb{R}$.

<u>EX</u>. LET $\mathbb{C}$ DENOTE THE SET OF COMPLEX NUMBERS, i.e.

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

WHERE i DENOTES THE <u>IMAGINARY UNIT</u>. ($i^2 = -1$.) DEFINE

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$
$$k(a+bi) = (ka) + (kb)i$$
$$0 = 0 + 0i$$
$$-(a+bi) = (-a) + (-b)i$$

AXIOMS 1-8 ARE VERIFIED EASILY. WE CAN IDENTIFY THIS EXAMPLE WITH THE PLANE $\mathbb{R}^2$ BY

$$a + bi \longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$

THIS CONCEPT OF A LINEAR SPACE UNIFIES MANY DIFFERENT EXAMPLES. ALL THE FAMILIAR NOTIONS WHICH WE DEVELOPED IN $\mathbb{R}^n$, ARE EQUALLY VALID IN ANY LINEAR SPACE. THESE INCLUDE

- LINEAR COMBINATIONS
- LINEAR TRANSFORMATIONS
- KERNAL, image, SUBSPACE
- LINEAR INDEPENDENCE, SPAN, BASIS

<u>DEFN</u>.
LET V BE A LINEAR SPACE AND $W \subseteq V$. WE CALL W A SUBSPACE OF V IFF

a.) $0 \in W$
b.) $f, g \in W \Rightarrow f + g \in W$
c.) $f \in W, k \in \mathbb{R} \Rightarrow kf \in W$

OBSERVE THAT A SUBSPACE OF A LINEAR SPACE IS ITSELF A LINEAR SPACE UNDER THE VERY SAME OPERATIONS.

<u>EX</u>. $F(\mathbb{R}, \mathbb{R})$ HAS MANY INTERESTING SUBSPACES. WE LIST A FEW HERE.

- $C^0(\mathbb{R}, \mathbb{R})$ IS THE SET OF ALL CONTINUOUS FUNCTIONS $\mathbb{R} \to \mathbb{R}$.

- $C^1(\mathbb{R},\mathbb{R})$ IS THE SET OF ALL FUNCTIONS IN $F(\mathbb{R},\mathbb{R})$ WHOSE 1^{ST} DERIVATIVE EXISTS AND IS CONTINUOUS.

- $C^k(\mathbb{R},\mathbb{R})$ WHERE $k \geq 0$ IS ANY INTEGER IS THE SET OF ALL k-TIMES CONTINUOUSLY DIFFERENTIABLE FUNCTIONS IN $F(\mathbb{R},\mathbb{R})$.

- $C^\infty(\mathbb{R},\mathbb{R})$ CONSISTS OF ALL INFINITELY DIFFERENTIABLE FUNCTIONS.

- P IS THE SET OF POLYNOMIAL FUNCTIONS

- P_n CONSISTS OF ALL POLYNOMIALS OF DEGREE AT MOST n.

- THE SET OF ALL SOLUTIONS TO A LINEAR, HOMOGENEOUS ODE (e.g. $f''(x)+2f'(x)-f(x)=0$). IS A SUBSPACE OF $F(\mathbb{R},\mathbb{R})$.

<u>EX.</u> LET $M_n = M_{nn}$ DENOTE THE SET OF ALL $n \times n$ SQUARE MATRICES. FIX $A \in M_n$. THEN

$$\{B \in M_n \mid AB = BA\}$$

IS A SUBSPACE OF M_n, AS IS EASILY VERIFIED (EXERCISE.)

<u>EX.</u> LET $GL(n) \subseteq M_n$ DENOTE THE SET OF ALL INVERTIBLE $n \times n$ MATRICES (CALLED THE <u>GENERAL LINEAR GROUP OF ORDER n</u>.)

THIS IMPORTANT SUBSET OF M_n IS <u>NOT</u> A SUBSPACE OF M_n. TO SEE THIS LET $n=2$ AND CONSIDER

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \quad \text{AND} \quad B = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$$

THEN $\det A = 5 \neq 0$ AND $\det B = 1 \neq 0$ WHENCE $A, B \in GL(2)$. BUT

$$A + B = \begin{pmatrix} 4 & 4 \\ 8 & 8 \end{pmatrix}$$

HENCE $\det(A+B) = 0$, SHOWING $A+B \notin GL(2)$.

A MORE BASIC REASON WHY $GL(n)$ IS NOT A SUBSPACE OF M_n IS SIMPLY THAT $0 \notin GL(n)$, i.e. THE ZERO MATRIX IS NOT INVERTIBLE.

<u>EX.</u> LET V BE A LINEAR SPACE AND $f_1, \dots, f_n \in V$. DEFINE

$$\operatorname{SPAN}(f_1, \dots, f_n) = \{ c_1 f_1 + \dots + c_n f_n \mid c_i \in \mathbb{R}, 1 \le i \le n \}$$

i.e. $\text{SPAN}(f_1, .., f_n) \subseteq V$ CONSISTS OF ALL LINEAR COMBINATIONS OF THE ELEMENTS $f_1, \dots, f_n$. THEN

- $0 = 0f_1 + \dots + 0f_n \in \text{SPAN}(f_1, .., f_n)$
- $(c_1f_1+\dots+c_nf_n)+(d_1f_1+\dots+d_nf_n)$
  $= (c_1+d_1)f_1 + \dots + (c_n+d_n)f_n \in \text{SPAN}(f_1, .., f_n)$
- $k(c_1f_1+\dots+c_nf_n) = (kc_1)f_1 + \dots + (kc_n)f_n \in \text{SPAN}(f_1, .., f_n)$

$\therefore$ $\text{SPAN}(f_1, .., f_n)$ IS A SUBSPACE OF V.

<u>DEFN</u>

LET V BE A LINEAR SPACE, AND $f_1, .., f_n \in V$.

(1) WE SAY $\{f_1, .., f_n\}$ <u>SPANS</u> V IFF $V = \text{SPAN}(f_1, \dots, f_n)$.

(2) WE SAY $\{f_1, .., f_n\}$ IS <u>LINEARLY INDEPENDENT</u> IFF THE EQUATION

$$c_1f_1 + \dots + c_nf_n = 0$$

IMPLIES

$$c_1 = \dots = c_n = 0$$

(3) WE SAY $\{f_1, .., f_n\}$ IS A <u>BASIS</u> OF V IFF IT IS BOTH LINEARLY INDEPENDENT AND SPANS V.

THEOREM

$\mathcal{B} = \{f_1, \ldots, f_n\} \subseteq V$ is a basis of V iff each $f \in V$ can be expressed uniquely as a linear combination of $f_1, \ldots, f_n$, i.e.

$$f = c_1 f_1 + \cdots + c_n f_n$$

for some unique coefficients $c_1, \ldots, c_n \in \mathbb{R}$.

The proof is identical to that given for $\mathbb{R}^n$, and we leave it as an exercise.

We call the column vector

$$\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$$

the <u>coordinates of f with respect to $\mathcal{B}$</u>, or just the <u>$\mathcal{B}$-coordinates of f</u>.

We write

$$[f]_{\mathcal{B}} = \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Define $T : V \to \mathbb{R}^n$ by $T(f) = [f]_{\mathcal{B}}$,

CALLED THE $\mathcal{B}$-COORDINATE TRANSFORMATION. T IS AN INVERTIBLE MAPPING WITH $T^{-1}: \mathbb{R}^n \to V$ GIVEN BY

$$T^{-1}(\vec{c}) = c_1 f_1 + \cdots + c_n f_n$$

THEOREM

LET $\mathcal{B}$ BE A BASIS OF A LINEAR SPACE V. THEN

$$\text{a.) } [f+g]_{\mathcal{B}} = [f]_{\mathcal{B}} + [g]_{\mathcal{B}}$$

AND

$$\text{b.) } [kf]_{\mathcal{B}} = k[f]_{\mathcal{B}}$$

FOR ALL $f, g \in V$ AND $k \in \mathbb{R}$.

THE PROOF IS ENTIRELY SIMILAR TO THE ONE GIVEN FOR $\mathbb{R}^n$, AND IS LEFT AS AN EXERCISE.

THUS THE $\mathcal{B}$-COORDINATE TRANSFORMATION $T: V \to \mathbb{R}^n$ HAS EVERY RIGHT TO BE CALLED A LINEAR TRANSFORMATION, ALTHOUGH WE HAVE NOT YET DEFINED THIS NOTION FOR GENERAL LINEAR SPACES. (SEE NEXT SECTION.)

THEOREM

IF A LINEAR SPACE V HAS A BASIS WITH n ELEMENTS, THEN ALL OTHER BASES OF V CONSIST OF n ELEMENTS ALSO. WE CALL n THE <u>DIMENSION</u> OF V: $\dim(V)$

PROOF.

LET $\mathcal{B}=\{f_1,\dots,f_n\}$ AND $\mathcal{C}=\{g_1,\dots,g_m\}$ BE TWO BASES FOR V. WE MUST SHOW THAT $m=n$.

CONSIDER THE VECTORS

$$[g_1]_{\mathcal{B}},\ \dots,\ [g_m]_{\mathcal{B}} \in \mathbb{R}^n$$

WE CLAIM THESE VECTORS ARE LINEARLY INDEPENDENT, FROM WHICH IT FOLLOWS THAT $m \le n$ (SEE P. 94 OF THESE NOTES.)

TO PROVE THE CLAIM, SUPPOSE

$$c_1[g_1]_{\mathcal{B}} + \cdots + c_m[g_m]_{\mathcal{B}} = \vec{0} \quad .$$

BY THE PRECEDING THEOREM

$$[c_1g_1 + \cdots + c_mg_m]_{\mathcal{B}} = \vec{0} \quad .$$

SINCE THE $\mathcal{B}$-COORDINATE TRANSFORMATION IS BIJECTIVE AND $[0]_{\mathcal{B}} = \vec{0}$, WE HAVE

$$c_1g_1 + \cdots + c_mg_m = 0 \quad .$$

BUT $\mathcal{C}$ BEING A BASIS, IS LINEARLY INDEPENDENT, HENCE $c_1 = \cdots = c_m = 0$.

WE CONCLUDE THAT $\{[g_1]_{\mathcal{B}},\dots,[g_m]_{\mathcal{B}}\} \subseteq \mathbb{R}^n$ IS LINEARLY INDEPENDENT, WHENCE $m \le n$.

REVERSING THE ROLES OF $\mathcal{B}$ AND $\mathcal{C}$ WE FIND THAT

$$\{[f_1]_{\mathcal{C}}, \ldots, [f_n]_{\mathcal{C}}\} \subseteq \mathbb{R}^m$$

IS ALSO LINEARLY INDEPENDENT, SO $n \leq m$.

IT FOLLOWS THAT $m=n$, AS REQUIRED.

///.

EX. RECALL $M_2 = \{2\times 2 \text{ MATRICES}\}$. LET

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

ONE CHECKS (EXERCISE) THAT $\mathcal{B}$ IS A BASIS OF M_2, SO $\dim(M_2) = 4$.

MORE GENERALLY $\dim(M_n) = n^2$ AND $\dim(M_{nm}) = nm$, WHERE

$$M_n = \{n\times n \text{ MATRICES}\}$$

$$M_{nm} = \{n\times m \text{ MATRICES}\}.$$

EX. LET V BE THE SUBSPACE OF M_2 CONSISTING OF ALL MATRICES THAT COMMUTE WITH

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

i.e. $V = \{ A \in M_2 \mid AB = BA \}$

TO FIND A BASIS FOR V, WE FIRST TRY TO CHARACTERIZE A TYPICAL ELEMENT OF V. LET

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V.$$

THEN

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\therefore \begin{pmatrix} a & a+2b \\ c & c+2d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 2c & 2d \end{pmatrix}$$

$$\therefore \begin{cases} a = a+c & \Rightarrow c = 0 \\ a+2b = b+d & \Rightarrow a+b = d \\ c = 2c & \\ c+2d = 2d & \end{cases}$$

THUS

$$A = \begin{pmatrix} a & b \\ 0 & a+b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

AND THEREFORE

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

SPANS V, AND IS EASILY SEEN TO BE LINEARLY INDEPENDENT (EXERCISE.) $\therefore \dim V = 2$.

Ex. $P_n = \{$ Polynomials of degree $\leq n\}$ has basis

$$\mathcal{B} = \{1, x, x^2, \ldots, x^n\}$$

(Exercise), hence $\dim(P_n) = n+1$.

Consider $V = \{ f \in P_n \mid f(0) = 0 \} \subseteq P_n$. Then V is a subspace of P_n with basis

$$\{x, x^2, \ldots, x^n\}$$

(Exercise), whence $\dim V = n$.

Note that not every linear space has a basis in the sense given on p. 137 (notes.) i.e. it is possible for there to be no finite set $\mathcal{B} \subseteq V$ which is both linearly independent, and spans V.

Such spaces are said to be <u>infinite dimensional</u>. Otherwise, if V contains a finite basis, it is called <u>finite dimensional</u>.

It is possible to give a useful definition of basis in the infinite dimensional case, but that definition is outside the scope of this course.

FINITE DIMENSIONAL SPACES:

$\mathbb{R}^n$, M_{nm}, P_n, {solns to lin. Hom. ODE}

INFINITE DIMENSIONAL SPACES:

$F(\mathbb{R},\mathbb{R})$, $C^0, C^1, \dots, C^k, \dots, C^\infty, C^\omega, P$

(4.1) 2, 6-24 even, 28, 30, 32, 34, 36