3.4 COORDINATES

LET $V \subseteq \mathbb{R}^n$ BE A SUBSPACE AND LET $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\} \subseteq V$. RECALL THAT $\mathcal{B}$ IS A BASIS OF V IFF EACH $\vec{x} \in V$ CAN BE WRITTEN UNIQUELY AS A LINEAR COMBINATION OF ELEMENTS OF $\mathcal{B}$, i.e. THERE EXIST UNIQUE COEFFICIENTS $c_1, \dots, c_m \in \mathbb{R}$ SUCH THAT

$$\vec{x} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$$

THIS EQUATION CAN BE WRITTEN AS

$$\vec{x} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

OR AS

$$\vec{x} = S\vec{c}$$

WHERE $S = [\vec{v}_1 \cdots \vec{v}_m]$ AND $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$.

<u>DEFN</u>

THE VECTOR $\vec{c} \in \mathbb{R}^m$ IS CALLED THE <u>COORDINATE VECTOR OF $\vec{x}$ WITH RESPECT TO $\mathcal{B}$</u> OR JUST THE <u>$\mathcal{B}$-COORDINATE VECTOR OF $\vec{x}$</u>. WE WRITE

$$[\vec{x}]_{\mathcal{B}} = \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbb{R}^m.$$

THUS, THE ABOVE EQUATIONS CAN BE EXPRESSED AS

*
$$\boxed{\vec{x} = S\,[\vec{x}]_{\beta}}$$

Ex. LET $V = \mathrm{SPAN}\left(\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right) \subseteq \mathbb{R}^3$. THEN V IS A PLANE THROUGH $\vec{0} \in \mathbb{R}^3$, AND OBVIOUSLY

$$\beta = \left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$$

FORMS A BASIS OF V. LET $\vec{x}$ BE THE VECTOR

$$\vec{x} = \begin{pmatrix}2\\5\\1\end{pmatrix} = 2\begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}0\\1\\-1\end{pmatrix} \in V$$

THEN $[\vec{x}]_{\beta} = \begin{pmatrix}2\\1\end{pmatrix} \in \mathbb{R}^2$. ONE CHECKS EASILY THAT

$$S[\vec{x}]_{\beta} = \begin{pmatrix}1 & 0\\2 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2\\5\\1\end{pmatrix} = \vec{x}$$

AS EXPECTED,

THEOREM

LET $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ BE A BASIS OF $V \subseteq \mathbb{R}^n$ AND LET $\vec{x}, \vec{y} \in V$, $k \in \mathbb{R}$. THEN

a.) $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$

b.) $[k\vec{x}]_{\mathcal{B}} = k\,[\vec{x}]_{\mathcal{B}}$

PROOF:

SINCE $\mathcal{B}$ IS A BASIS, THERE EXIST UNIQUE $c_1, \ldots, c_m, d_1, \ldots, d_m$ SUCH THAT

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$$

AND

$$\vec{y} = d_1\vec{v}_1 + \cdots + d_m\vec{v}_m$$

SO $$\vec{x} + \vec{y} = (c_1 + d_1)\vec{v}_1 + \cdots + (c_m + d_m)\vec{v}_m .$$

BUT THESE EQUATIONS SAY

$$[\vec{x} + \vec{y}]_{\mathcal{B}} = \begin{pmatrix} c_1 + d_1 \\ \vdots \\ c_m + d_m \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} + \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}$$

$$= [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}} ,$$

PROVING (a.). WE LEAVE (b) AS AN EXERCISE.

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A SPECIAL CASE OCCURS WHEN $m = n$, i.e. WHEN $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ IS A BASIS FOR $\mathbb{R}^n$ ITSELF. IN THIS CASE

$$S = [\vec{v}_1 \dots \vec{v}_n]$$

IS AN INVERTIBLE $n \times n$ MATRIX, AND EQUATION $*$ CAN BE WRITTEN

$$** \quad \boxed{[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}}$$

<u>EX.</u> FIND THE COORDINATES OF $\begin{pmatrix} 5 \\ 7 \end{pmatrix} \in \mathbb{R}^2$ WITH RESPECT TO THE BASIS

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

IN THIS CASE

$$S^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

AND

$$\left[\begin{pmatrix} 5 \\ 7 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

INDEED, ONE CHECKS

$$\begin{pmatrix} 5 \\ 7 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

THUS EACH BASIS $\mathcal{B}$ DEFINES ITS OWN COORDINATE SYSTEM IN $\mathbb{R}^n$. EVERY POINT (VECTOR) OTHER THAN $\vec{0}$, HAS A DIFFERENT NAME IN EACH COORDINATE SYSTEM. Similarly, linear TRANSFORMATIONS ARE EXPRESSED DIFFERENTLY IN DIFFERENT COORDINATE SYSTEMS.

LET $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ BE A BASIS OF $\mathbb{R}^n$ AND SUPPOSE

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

is A linear TRANSFORMATION (NOT NECESSARily INVERTIBLE, AND SUPPOSE A is THE $(n \times n)$ MATRIX FOR T.

LET $F : \mathbb{R}^n \to \mathbb{R}^n$ BE THE FUNCTION WHICH MAPS THE $\mathcal{B}$-COORDINATES OF $\vec{x} \in \mathbb{R}^n$ TO THE $\mathcal{B}$-COORDINATES OF $T(\vec{x}) = A\vec{x}$.

i.e
$$F([x]_{\mathcal{B}}) = [T(\vec{x})]_{\mathcal{B}}$$

FIRST OBSERVE THAT F IS A LINEAR TRANSFORMATION. INDEED,

$$\begin{aligned} F([\vec{x}]_{\mathcal{B}} + [y]_{\mathcal{B}}) &= F([x+y]_{\mathcal{B}}) \\ &= [T(x+y)]_{\mathcal{B}} \\ &= [T(x) + T(y)]_{\mathcal{B}} \\ &= [T(x)]_{\mathcal{B}} + [T(y)]_{\mathcal{B}} \\ &= F([x]_{\mathcal{B}}) + F([y]_{\mathcal{B}}), \end{aligned}$$

AND WE LEAVE IT AS AN EXERCISE TO SHOW

$$F(k[x]_{\mathcal{B}}) = k F([x]_{\mathcal{B}})$$

LET B BE THE $n \times n$ MATRIX FOR F, i.e. B IS DEFINED BY THE EQUATION

$$B[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{B}}.$$

WE CALL B THE <u>MATRIX OF T WITH RESPECT TO $\mathcal{B}$</u>, OR SIMPLY THE <u>$\mathcal{B}$-MATRIX OF T</u>.

THIS EQUATION CAN ALSO BE WRITTEN

$$B[\vec{x}]_{\mathcal{B}} = [A\vec{x}]_{\mathcal{B}}$$

NOW RECALL SINCE $\{\vec{v}_1, \ldots, \vec{v}_n\}$ IS A BASIS, $S = [\vec{v}_1 \cdots \vec{v}_n]$ IS INVERTIBLE, AND

$$[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x} \quad \text{FOR ALL } \vec{x} \in \mathbb{R}^n.$$

IN PARTICULAR

$$[A\vec{x}]_{\mathcal{B}} = S^{-1}A\vec{x} \quad .$$

THE DEFINING EQUATION FOR B IS NOW

$$BS^{-1}\vec{x} = S^{-1}A\vec{x} \quad .$$

SINCE THIS HOLDS FOR ALL $\vec{x} \in \mathbb{R}^n$, IT FOLLOWS THAT $BS^{-1} = S^{-1}A$, WHENCE

$$\boxed{B = S^{-1}AS}$$

OR EQUIVANTLY

$$SB = AS.$$

WE'VE PROVED:

THEOREM

LET $T: \mathbb{R}^n \to \mathbb{R}^n$ BE LINEAR WITH MATRIX A, AND $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ A BASIS OF $\mathbb{R}^n$. THEN THE MATRIX OF T WITH RESPECT TO $\mathcal{B}$ IS

$$B = S^{-1} A S$$

WHERE $S = [\vec{v}_1 \; \cdots \; \vec{v}_n]$.

NOTE THAT A IS THE MATRIX OF T WITH RESPECT TO THE STANDARD BASIS $\{\vec{e}_1, \ldots, \vec{e}_n\}$ SINCE $I_n = [\vec{e}_1 \cdots \vec{e}_n]$, AND $I_n^{-1} A I_n = A$.

EX. LET $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. FIND THE MATRIX OF $T_A: \mathbb{R}^2 \to \mathbb{R}^2$ WITH RESPECT TO THE BASIS

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\},$$

AND VERIFY THAT

$$B[\vec{x}]_{\mathcal{B}} = [A\vec{x}]_{\mathcal{B}}$$

FOR ALL $\vec{x} \in \mathbb{R}^2$.

WE HAVE $S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ AND $S = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$
SO THE MATRIX B OF T_A IS

$$B = S^{-1} A S = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 4 & 6 \end{pmatrix}.$$

NOW PICK ANY $\vec{x} \in \mathbb{R}^2$. THEN

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_1 - x_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-x_1 + x_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

SO THAT

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix}.$$

ALSO

$$A\vec{x} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$$

$$= (-x_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (2x_1 + 2x_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

SO THAT

$$[A\vec{x}]_{\mathcal{B}} = \begin{pmatrix} -x_1 \\ 2x_1 + 2x_2 \end{pmatrix}$$

THEREFORE

$$B\,[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} -1 & -1 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix}$$

$$= \begin{pmatrix} (-2x_1 + x_2) + (x_1 - x_2) \\ (8x_1 - 4x_2) + (-6x_1 + 6x_2) \end{pmatrix}$$

$$= \begin{pmatrix} -x_1 \\ 2x_1 + 2x_2 \end{pmatrix}$$

$$= [A\vec{x}]_{\mathcal{B}} .$$

LET US GIVE ANOTHER PROOF OF THE PRECEDING THEOREM. LET

$$\vec{c} = [\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x} \qquad (\text{BY } **)$$

SO THAT

$$F(\vec{c}) = F([\vec{x}]_{\mathcal{B}}) = [A\vec{x}]_{\mathcal{B}}$$

$$= [AS\vec{c}]_{\mathcal{B}}$$

$$= S^{-1}AS\vec{c} \qquad (\text{AGAIN BY } **).$$

SINCE THIS HOLDS FOR ALL $\vec{c} \in \mathbb{R}^n$, THE MATRIX OF F IS $S^{-1}AS$, AS EXPECTED.

OBSERVE THAT THE j^{TH} COLUMN OF $B = S^{-1}AS$ IS

$$B\vec{e}_j = S^{-1}A(S e_j) = S^{-1}A\vec{v}_j$$
$$= [A\vec{v}_j]_{\mathcal{B}} = [T_A(\vec{v}_j)]_{\mathcal{B}}.$$

THUS ANOTHER CHARACTERIZATION OF B IS

$$B = \left[[T_A(\vec{v}_1)]_{\mathcal{B}} \quad \cdots \quad [T_A(\vec{v}_n)]_{\mathcal{B}} \right].$$

DEFN.

LET A, B BE $n \times n$ MATRICES. WE SAY THAT A IS SIMILAR TO B IFF THERE EXISTS AN INVERTIBLE $n \times n$ MATRIX S SUCH THAT

$$AS = SB \quad \text{i.e.} \quad B = S^{-1}AS$$

THEOREM:

A IS SIMILAR TO B IFF B IS THE MATRIX OF T_A WITH RESPECT TO SOME BASIS OF $\mathbb{R}^n$.

PROOF:

($\Leftarrow$) THIS IS JUST THE PRECEDING THEOREM.

($\Rightarrow$) EXERCISE. ///

WE WRITE

$$A \sim B$$

TO MEAN THAT A IS SIMILAR TO B. OBSERVE THAT THE SIMILARITY RELATION SATISFIES THE FOLLOWING PROPERTIES.

a.) REFLEXIVE. $A \sim A$ SINCE $A = I_n^{-1} A I_n$.

b.) SYMMETRIC. IF $A \sim B$, THEN $B \sim A$ SINCE

$$B = S^{-1} A S \implies A = (S^{-1})^{-1} B (S^{-1}).$$

c.) TRANSITIVE. IF $A \sim B$ AND $B \sim C$, THEN $A \sim C$.

Pf. SUPPOSE $B = S_1^{-1} A S_1$ AND $C = S_2^{-1} B S_2$. THEN

$$C = S_2^{-1}(S_1^{-1} A S_1) S_2 = (S_1 S_2)^{-1} A (S_1 S_2)$$

SINCE $S_1 S_2$ IS INVERTIBLE WITH $(S_1 S_2)^{-1} = S_2^{-1} S_1^{-1}$. HENCE $A \sim C$.

///.

A RELATION SATISFYING PROPERTIES (a), (b), AND (c) ABOVE IS CALLED AN <u>EQUIVALENCE RELATION</u>.

<u>EXERCISE</u>.
LET $A \sim B$ AND LET $p \geq 0$ BE AN INTEGER. SHOW THAT

$$A^p \sim B^p.$$

<u>EXERCISE</u>.
LET $A \sim B$. SHOW THAT

$$\operatorname{rank}(A) = \operatorname{rank}(B).$$

(<u>HINT</u>: LET $B = S^{-1}AS$, $r = \operatorname{rank}(B)$, AND SUPPOSE $\{\vec{u}_1, \dots, \vec{u}_r\}$ IS A BASIS FOR $\operatorname{im}(B)$. SHOW THAT $\{S\vec{u}_1, \dots, S\vec{u}_r\}$ IS A BASIS FOR $\operatorname{im}(A)$, WHENCE $\operatorname{rank}(A) = \dim(\operatorname{im} A) = r = \operatorname{rank}(B)$.)

HW (3.4): 6-30 even, 40, 52, 56, 58, 60, 62, 64