3.3 DIMENSION OF A SUBSPACE OF $\mathbb{R}^n$

LET $V \subseteq \mathbb{R}^n$ BE A SUBSPACE, AND CONSIDER SEPARATELY THE TWO PROPERTIES THAT DEFINE A BASIS $\{\vec{v}_1, \dots, \vec{v}_m\}$ OF V.

(i) $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$
(ii) $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent

OBSERVE THAT ANY SUPERSET OF A SPANNING SET (i.e. ONE SATISFYING (i)) IS ALSO A SPANNING SET, WHILE A SUBSET OF SUCH A SET MAY NOT SATISFY (i).

ON THE OTHER HAND, ANY SUBSET OF A LINEARLY INDEPENDENT SET IS LINEARLY INDEPENDENT, WHILE A SUPERSET OF SUCH A SET MAY NOT HAVE THAT PROPERTY.

AS WE SHALL SEE, A BASIS FOR V IS SIMULTANEOUSLY A SMALLEST POSSIBLE SPANNING SET, AND A LARGEST POSSIBLE LINEARLY INDEPENDENT SET IN V.

<u>THEOREM</u>
LET $V \subseteq \mathbb{R}^n$ BE A SUBSPACE AND LET $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q \in V$. SUPPOSE THAT BOTH

- $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent

AND

- $\{\vec{w}_1, \dots, \vec{w}_q\}$ spans V.

THEN $p \le q$.

PROOF:

DEFINE MATRICES $A = [\vec{w}_1 \cdots w_q]$ $(n \times q)$, AND $B = [\vec{v}_1 \cdots \vec{v}_p]$ $(n \times p)$. THEN

$$\operatorname{im}(A) = \operatorname{span}(\vec{w}_1, \dots, \vec{w}_q) = V,$$

WHENCE $\vec{v}_1, \dots, \vec{v}_p \in \operatorname{im}(A)$. THEREFORE THERE EXIST $\vec{u}_1, \dots, \vec{u}_p \in \mathbb{R}^q$ SUCH THAT

$$A\vec{u}_1 = \vec{v}_1, \quad \dots\dots\dots, \quad A\vec{u}_p = \vec{v}_p \quad .$$

THUS

$$B = [\vec{v}_1 \cdots \vec{v}_p] = [A\vec{u}_1 \cdots A\vec{u}_p] = A[\vec{u}_1 \cdots \vec{u}_p].$$

NOW LET $C = [\vec{u}_1 \cdots \vec{u}_p]$ $(q \times p)$, SO THE LAST EQUATION IS

$$B = AC.$$

NOW OBSERVE THAT $\ker(C) \subseteq \ker(B)$, FOR IF $C\vec{x} = \vec{0}$, THEN $B\vec{x} = AC\vec{x} = A\vec{0} = \vec{0}$.

BUT $\mathrm{KER}(B)=\{\vec{0}\}$ SINCE $\{\vec{v}_1,\ldots,\vec{v}_p\}$ is linearly independent, whence also $\mathrm{KER}(C)=\{\vec{0}\}$. But as we've already seen, this is equivalent to

$$\mathrm{rank}(C) = p \leq q ,$$

i.e. $p \leq q$, as claimed. ///

<u>Corollary</u>
Let $V \subseteq \mathbb{R}^n$ be a subspace. Then any two bases for V have the same cardinality.

<u>Proof</u>:
Let $\{\vec{v}_1,\ldots,\vec{v}_p\}$ and $\{\vec{w}_1,\ldots,\vec{w}_q\}$ be two bases for V. By the preceding theorem $p \leq q$ since $\{\vec{v}_1 \cdots \vec{v}_p\}$ is linearly independent, and $\{\vec{w}_1 \cdots \vec{w}_q\}$ spans V. But also $\{\vec{w}_1 \cdots \vec{w}_q\}$ is linearly independent and $\{\vec{v}_1 \cdots \vec{v}_p\}$ spans V, whence $q \leq p$ by the same theorem. Thus $q = p$.

///

DEFN

THE DIMENSION OF A SUBSPACE $V \subseteq \mathbb{R}^n$ IS THE NUMBER OF VECTORS IN ANY BASIS FOR V. WE WRITE $\dim(V)$ FOR THIS QUANTITY.

NECESSARILY $\dim(V) \leq n$ BY ITEM (6) ON P. 95 OF THESE NOTES.

OBSERVE THAT $\mathbb{R}^n$ IS ITSELF A SUBSPACE OF $\mathbb{R}^n$, AS ONE MAY EASILY VERIFY. WHAT IS ITS DIMENSION? RECALL THE STANDARD BASIS $\{\vec{e}_1, \dots, \vec{e}_n\} \subseteq \mathbb{R}^n$ DEFINED BY

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{TH}} \text{ ROW} \qquad (1 \leq i \leq n)$$

EXERCISE

PROVE THAT $\{\vec{e}_1, \dots \vec{e}_n\}$ SPANS $\mathbb{R}^n$, AND IS LINEARLY INDEPENDENT, AND HENCE IT IS INDEED A BASIS OF $\mathbb{R}^n$.

THUS $\dim(\mathbb{R}^n) = n$, AS ONE WOULD EXPECT.

OBSERVE ALSO THAT $\{\vec{0}\} \subseteq \mathbb{R}^n$ IS A SUBSPACE (CHECK.) IT IS SOMETIMES TAKEN AS CONVENTION THAT

$$\dim\{\vec{0}\} = 0 .$$

AN ALTERNATE (AND PERHAPS MORE SATISFYING) APPROACH IS TO REGARD $\vec{0}$ AS AN EMPTY SUM, i.e. ONE HAVING NO TERMS, SO THAT $\emptyset \subseteq \{\vec{0}\}$ SATISFIES THE PROPERTIES OF A BASIS (CHECK).

<u>THEOREM</u>
LET $\dim(V) = m$, $\vec{v}_1 \cdots \vec{v}_p, \vec{w}_1 \cdots \vec{w}_q, \vec{u}_1 \cdots \vec{u}_m \in V$, AND SUPPOSE:

$\{\vec{v}_1 \cdots \vec{v}_p\}$ is LINEARLY INDEPENDENT,
$\{\vec{w}_1 \cdots \vec{w}_q\}$ SPANS V,
$\{\vec{u}_1 \cdots \vec{u}_m\}$ is A BASIS FOR V.

THEN
(a) $p \leq m \leq q$
(b) $p = m \Rightarrow \{\vec{v}_1 \cdots \vec{v}_p\}$ is A BASIS OF V
(c) $q = m \Rightarrow \{\vec{w}_1 \cdots \vec{w}_q\}$ is A BASIS OF V

PROOF:

(a) FOLLOWS INSTANTLY FROM THE PREVIOUS THEOREM, AND THE BOOK DOES (b). WE PROVE (c) NOW.

WE SHOW THAT IF $\{\vec{w}_1 \cdots \vec{w}_q\}$ IS LINEARLY DEPENDENT, THEN $m < q$, WHICH IS THE CONTRAPOSITIVE OF THE IMPLICATION IN (c).

NOW IF $\{\vec{w}_1, \ldots, \vec{w}_q\}$ IS LINEARLY DEPENDENT THEN ONE OF ITS VECTORS IS REDUNDANT, SAY FOR DEFINITENESS THAT $\vec{w}_q$ IS REDUNDANT IN THE LIST $\vec{w}_1 \cdots w_q$. THUS $\{\vec{w}_1, \ldots, w_{q-1}\}$ IS ALSO A SPANNING SET IN V. BUT THEN BY (a) WE HAVE

$$m \leq q-1 < q \quad ,$$

AS REQUIRED. ///

WE NOW SEEK A SYSTEMATIC APPROACH TO SOLVING THE FOLLOWING PROBLEM.

GIVEN AN $n \times m$ MATRIX A, DETERMINE BOTH

- A BASIS OF $\mathrm{im}(A) \subseteq \mathbb{R}^n$
- A BASIS OF $\mathrm{KER}(A) \subseteq \mathbb{R}^m$

AS WE WILL SEE THESE TWO SEEMINGLY DIFFERENT PROBLEMS ARE CLOSELY RELATED, AND ARE IN FACT ONE.

FIRST LET $B = \mathrm{RREF}(A)$ AND OBSERVE THAT

$$\mathrm{KER}(A) = \mathrm{KER}(B).$$

INDEED, THE LINEAR SYSTEMS $(A \mid \vec{0})$ AND $(B \mid \vec{0})$ HAVE THE VERY SAME SOLUTIONS BY THE NATURE OF THE ELEMENTARY ROW OPERATIONS.

NEXT RECALL THAT IF

$$A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} \quad \text{AND} \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$$

THEN

$$A\vec{x} = x_1 \vec{v}_1 + \cdots\cdots + x_m \vec{v}_m .$$

THUS $A\vec{x}=\vec{0}$ IFF $x_1\vec{v}_1+\cdots+x_m\vec{v}_m=\vec{0}$.
IN OTHER WORDS ANY NON-ZERO ELEMENT OF KER(A) GIVES RISE TO A NON-TRIVIAL RELATION AMONGST THE COLUMNS OF A, AND CONVERSELEY. THE SAME GOES FOR B, AND SINCE $\mathrm{KER}(A)=\mathrm{KER}(B)$, WE HAVE THE FOLLOWING VERY USEFUL FACT

> ANY NON-TRIVIAL RELATION AMONGST THE COLUMNS OF A ALSO EXISTS AMONGST THE COLUMNS OF B (i.e, WITH THE SAME COEFFICIENTS), AND CONVERSELEY.

IN PARTICULAR, WE HAVE

> A GIVEN COLUMN OF A IS REDUNDANT IFF THE CORRESPONDING COLUMN OF B IS REDUNDANT.

BUT IT IS EASY TO SPOT REDUNDANT COLUMNS IN B SINCE IT IS IN RREF. THESE ARE PRECISELY THE COLUMNS NOT CONTAINING A LEADING 1, i.e. THOSE CORRESPONDING TO FREE VARIABLES OF THE SYSTEM $(B\,|\,\vec{0})$

THUS TO FIND A BASIS FOR $\mathrm{Im}(A)$, CALCULATE $B = \mathrm{RREF}(A)$, THEN LOCATE THE COLUMNS OF B CONTAINING A LEADING 1. THE CORRESPONDING COLUMNS IN A WILL FORM A BASIS FOR $\mathrm{Im}(A)$.

EX.

$$A = \begin{array}{c} \begin{pmatrix} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 3 & 4 & -6 & 8 & 1 \\ 0 & -1 & 3 & 1 & 2 \end{pmatrix} \\ \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{matrix} \end{array} \qquad (4 \times 5)$$

$$B = \mathrm{RREF}(A) = \begin{array}{c} \begin{pmatrix} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 & u_5 \end{matrix} \end{array}$$

OBSERVE:

$$u_3 = 2\vec{u}_1 - 3\vec{u}_2 \qquad \text{and} \qquad \vec{u}_4 = 4\vec{u}_1 - \vec{u}_2$$

$$\therefore \quad -2\vec{u}_1 + 3\vec{u}_2 + \vec{u}_3 = \vec{0} \qquad -4\vec{u}_1 + \vec{u}_2 + \vec{u}_4 = \vec{0}$$

$$(*) \quad \therefore \quad -2\vec{v}_1 + 3\vec{v}_2 + \vec{v}_3 = \vec{0} \qquad -4\vec{v}_1 + \vec{v}_2 + \vec{v}_4 = \vec{0}$$

$$\therefore \quad \vec{v}_3 = 2\vec{v}_1 - 3\vec{v}_2 \qquad \vec{v}_4 = 4\vec{v}_1 - \vec{v}_2$$

THUS $\vec{v}_3$ AND $\vec{v}_4$ ARE REDUNDANT IN THE LIST $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5$, AND HENCE

$\{\vec{v}_1, \vec{v}_2, \vec{v}_5\}$ SPANS $\text{im}(A)$. ALSO $\{\vec{u}_1, \vec{u}_2, \vec{u}_5\}$ IS OBVIOUSLY LINEARLY INDEPENDANT, AND SO ALSO IS $\{\vec{v}_1, \vec{v}_2, \vec{v}_5\}$.

THUS $\{\vec{v}_1, \vec{v}_2, \vec{v}_5\}$ FORMS A BASIS FOR $\text{im}(A)$.

NOTE ALSO THAT RELATIONS (*) ESTABLISH THAT

$$\left\{ \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subseteq \ker(A).$$

INDEED THE AUGMENTED SYSTEM $(B \mid \vec{0})$ GIVES THAT $\vec{x} \in \ker(A)$ IFF

$$\vec{x} = \begin{pmatrix} -2s-4t \\ 3s+t \\ s \\ t \\ 0 \end{pmatrix} = s \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

SO THAT THE ABOVE SET SPANS $\ker(A)$. THE SET IS ALSO LINEARLY INDEPENDENT, HENCE IS A BASIS FOR $\ker(A)$.

THIS PROCESS ALWAYS YIELDS A LINEARLY INDEPENDENT SET SINCE EACH VECTOR WILL HAVE A 1 IN SOME ROW WHERE THE OTHER VECTORS HAVE 0.

<u>EX</u> CONSIDER THE AUGMENTED SYSTEM IN RREF

$$(B|\vec{0}) = \begin{array}{c} \begin{matrix} * & s_1 & s_2 & * & * & s_3 & s_4 & * & s_5 & \end{matrix} \\ \left(\begin{array}{ccccccccc|c} 1 & 2 & 3 & 0 & 0 & 4 & 7 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 & 0 & 5 & 8 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & 1 & 6 & 9 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 13 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

THUS $\vec{x} \in \text{KER}(B) = \text{KER}(A)$ IFF

$$\vec{x} = \begin{pmatrix} -2s_1 - 3s_2 - 4s_3 - 7s_4 - 10s_5 \\ s_1 \\ s_2 \\ -5s_3 - 8s_4 - 11s_5 \\ -6s_3 - 9s_4 - 12s_5 \\ s_3 \\ s_4 \\ -13s_5 \\ s_5 \end{pmatrix}.$$

WHERE $s_1, s_2, s_3, s_4, s_5 \in \mathbb{R}$ ARE ARBITRARY.

A BASIS FOR $\text{KER}(A)$ IS THEREFORE GIVEN BY THE SET

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ -5 \\ -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 0 \\ 0 \\ -8 \\ -9 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -10 \\ 0 \\ 0 \\ -11 \\ -12 \\ 0 \\ 0 \\ -13 \\ 1 \end{pmatrix} \right\} \subseteq \mathrm{KER}(A)$$

THUS TO OBTAIN A BASIS FOR THE KERNAL OF AN $n \times m$ MATRIX A, DO THE FOLLOWING

- COMPUTE $B = \mathrm{RREF}(A)$ BY GAUSS-JORDAN ELIMINATION
- MARK THE COLUMNS OF B THAT CONTAIN A LEADING 1, AND LET $r = \#$ LEADING 1s.
- LET $p = m - r$, AND ASSIGN "FREE" VARIABLES $s_1, \ldots, s_p$ TO THE REMAINING COLUMNS
- FORM AN $m \times p$ MATRIX C BY PERFORMING THE FOLLOWING ALGORITHM.

```
1.) j ← 1, k ← 1
2.) for i ← 1 to m
3.)      if B contains a leading 1 in col i
4.)           set row i of C to be (j-1) 0s,
              followed by the negatives of the
              (p-j) free entries in row k of B
              to the right of that 1.
5.)           k ← k+1
6.)      else
7.)           set row i of C to be (j-1) 0s,
              followed by a 1 in col j, followed
              by (p-j) 0s.
8.)           j ← j+1
```

Once this is done, the columns of C form a basis for $\ker(A)$.

Indeed the columns of C are linearly independent since they each contain a 1 (from line 7) that is the sole non-zero entry in its row. Hence no column can be obtained as a linear combination of the other columns.

The columns of C also span $\ker(A)$, for if $\vec{x} \in \ker(A) = \ker(B)$, we can form a vector

$$\vec{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_p \end{pmatrix} \in \mathbb{R}^p$$

whose components $s_1, \ldots, s_p$ are precisely the components of $\vec{x}$ corresponding to free columns of B. One checks that

$$\vec{x} = C\vec{s}$$

whence $\vec{x} \in \mathrm{im}(C) = \mathrm{span}(\text{columns of } C)$.

Exercise

Carry out the above algorithm on the last example

These examples have demonstrated that

- $\dim(\mathrm{im}\, A) = \mathrm{rank}(A) = \#\text{ lead 1s in } B$
- $\dim(\ker A) = m - \mathrm{rank}(A) = \#\text{ free cols. in } B$

Defn

The nullity of A is the dimension of $\ker(A)$, denoted

$$\mathrm{nullity}(A) = \dim(\ker A).$$

Theorem (Rank-Nullity)

Let A be an $n \times m$ matrix. Then

$$\dim(\mathrm{im}\, A) + \dim(\ker A) = m$$

i.e.

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = m.$$

THEOREM

A SET $\{\vec{v}_1, \dots, \vec{v}_n\} \subseteq \mathbb{R}^n$ is A BASIS OF $\mathbb{R}^n$ IFF THE MATRIX

$$A = \left[\vec{v}_1 \cdots \vec{v}_n \right] \quad (n \times n)$$

is INVERTIBLE

PROOF:
RECALL $\{\vec{v}_1, \dots, \vec{v}_n\}$ is A BASIS OF $\mathbb{R}^n$ IFF IT is linearly INDEPENDENT, SINCE $\dim(\mathbb{R}^n) = n$. BUT AS WE'VE SEEN THIS is EQUIVALENT TO $\ker(A) = \{\vec{0}\}$, WHICH is EQUIVALENT TO A BEING INVERTIBLE. ///

THEOREM

LET A BE AN $n \times n$ MATRIX. THEN T.F.A.E.

(1) A is INVERTIBLE
(2) $A\vec{x} = \vec{b}$ HAS A UNIQUE SOLUTION FOR ALL $\vec{b} \in \mathbb{R}^n$
(3) $\mathrm{RREF}(A) = I_n$
(4) $\mathrm{rank}(A) = n$
(5) $\mathrm{im}(A) = \mathbb{R}^n$
(6) $\mathrm{nullity}(A) = 0$
(7) $\ker(A) = \{\vec{0}\}$
(8) THE COLUMNS OF A FORM A BASIS OF $\mathbb{R}^n$
(9) THE COLUMNS OF A SPAN $\mathbb{R}^n$
(10) THE COLUMNS OF A ARE LINEARLY INDEPENDENT.

HW (3.3): 6–32 even, 36, 38ab, 40, 42, 51a, 54, 64