3.2 SUBSPACES OF $\mathbb{R}^n$

DEFN

LET $W \subseteq \mathbb{R}^n$. WE CALL W A (LINEAR) SUBSPACE OF $\mathbb{R}^n$ IFF

a.) $\vec{0} \in W$
b.) IF $\vec{w}_1, \vec{w}_2 \in W$, THEN $\vec{w}_1 + \vec{w}_2 \in W$.
c.) IF $\vec{w} \in W$ AND $k \in \mathbb{R}$, THEN $k\vec{w} \in W$.

EX.

LET $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ AND $W = \text{SPAN}(\vec{v}_1, \ldots, \vec{v}_m)$. THEN W IS A SUBSPACE OF $\mathbb{R}^n$

CHECK

(a) $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_m \in W$.
(b) SUPPOSE $\vec{w}_1, \vec{w}_2 \in W$, SO THAT

$$\vec{w}_1 = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

AND $$\vec{w}_2 = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_m\vec{v}_n \quad .$$

$$\therefore \vec{w}_1 + \vec{w}_2 = (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 + \cdots + (c_m+d_m)\vec{v}_m \in W$$

(c) SUPPOSE $\vec{w} \in W$ AND $k \in \mathbb{R}$. THEN

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

$$\therefore k\vec{w} = (kc_1)\vec{v}_1 + (kc_2)\vec{v}_2 + \cdots + (kc_m)\vec{v}_m \in W.$$

THIS EXAMPLE IS CANNONICAL IN THE SENSE THAT EVERY SUBSPACE OF $\mathbb{R}^n$ IS THE SPAN OF A FINITE COLLECTION OF VECTORS. PROOF LATER.

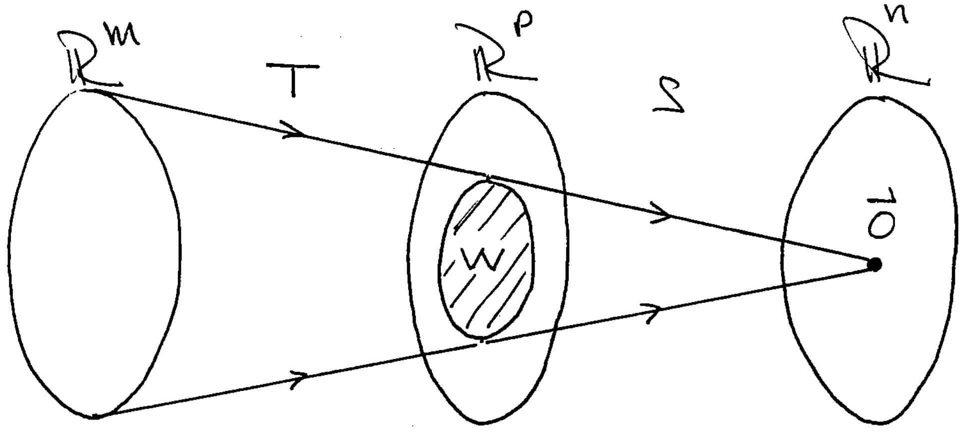
LET $T : \mathbb{R}^m \to \mathbb{R}^n$ BE LINEAR. WE HAVE SEEN ALREADY THAT

- $\operatorname{im}(T)$ IS A SUBSPACE OF $\mathbb{R}^n$

AND

- $\operatorname{KER}(T)$ IS A SUBSPACE OF $\mathbb{R}^m$.

WE WILL SEE THAT THIS CHARACTERIZATION OF SUBSPACES IS ALSO CANONICAL, i.e. EVERY SUBSPACE OF $\mathbb{R}^n$ IS THE IMAGE OF SOME LINEAR TRANSFORMATION, AND ALSO THE KERNEL OF SOME OTHER LINEAR TRANSFORMATION.



$$\operatorname{im}(T) = W = \operatorname{KER}(S)$$

<u>EX.</u>

LET P BE THE PLANE IN $\mathbb{R}^3$ DEFINED BY THE EQUATION

$$P : \; 2x_1 + x_2 + 5x_2 = 0.$$

OBSERVE THAT P IS INDEED A SUBSPACE,

SINCE IT IS THE KERNAL OF A CERTAIN 1X3 MATRIX, NAMELY

$$A = \begin{pmatrix} 2 & 1 & 5 \end{pmatrix}$$

OBVIOUSLY $\vec{x} \in P$ IFF $A\vec{x} = \vec{0}$, WHENCE

$$P = \ker(A).$$

TO EXHIBIT P AS AN IMAGE ONE NEED ONLY FIND TWO NON-PARALLEL VECTORS IN P, THEN FORM THE MATRIX WITH THOSE COLUMNS. FOR INSTANCE

$$B = \begin{pmatrix} 1 & -1 \\ -7 & 2 \\ 1 & 0 \end{pmatrix}$$

WILL DO. SINCE EACH COLUMN IS A SOLUTION WE HAVE

$$\operatorname{im} B = \operatorname{SPAN}\left(\begin{pmatrix} 1 \\ -7 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right) \subseteq P$$

TO PROVE THE OPPOSITE INCLUSION WE FIND ALL SOLUTIONS TO THE DEFINING EQUATION FOR P. LET $x_1 = t$, $x_3 = s$ BE FREE, THEN THE GENERAL SOLUTION IS

$$\vec{x} = \begin{pmatrix} t \\ -2t-5s \\ s \end{pmatrix} = s\begin{pmatrix} 1 \\ -7 \\ 1 \end{pmatrix} + (s-t)\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \in \operatorname{im} B$$

WHENCE $P \subseteq \mathrm{im}(B)$. $\therefore P = \mathrm{im}(B)$.

DEFN.
LET $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$.

- WE SAY A VECTOR $\vec{v}_i$ IN THIS LIST IS REDUNDANT IFF IT IS A LINEAR COMBINATION OF THE PRECEDING VECTORS, i.e.

$$\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1}$$

FOR SOME $c_1, \ldots, c_{i-1} \in \mathbb{R}$.

- WE SAY THE SET $\{\vec{v}_1, \ldots, \vec{v}_m\}$ IS LINEARLY INDEPENDENT IFF IT CONTAINS NO REDUNDANT VECTORS. OTHERWISE, THE SET IS CALLED LINEARLY DEPENDENT

- LET $V \subseteq \mathbb{R}^n$ BE A SUBSPACE. THE SET $\{\vec{v}_1, \ldots, \vec{v}_m\}$ IS CALLED A BASIS FOR V IFF

(i) $\{\vec{v}_1, \ldots, \vec{v}_m\}$ IS LINEARLY INDEPENDENT
AND (ii) $V = \mathrm{SPAN}(\vec{v}_1, \ldots, \vec{v}_m)$!

RMKS
1.) THE DEFINITION OF REDUNDANT VECTOR WOULD SEEM TO DEPEND ON THE ORDER IN WHICH WE LIST THE VECTORS.

INDEED, $\begin{pmatrix}1\\1\end{pmatrix}$ IS REDUNDANT IN THE list $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$, BUT NOT REDUNDANT IN $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$.

2.) HOWEVER, LINEAR INDEPENDENCE DOES NOT DEPEND ON ORDER, i.e. IT IS REALLY A PROPERTY OF THE SET $\{\vec{v}_1, .., \vec{v}_m\}$. SO ALTHOUGH $\begin{pmatrix}1\\1\end{pmatrix}$ IS NOT REDUNDANT IN $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, THE VECTOR $\begin{pmatrix}0\\1\end{pmatrix}$ IS REDUNDANT. THUS $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\}$ CONTAINS A REDUNDANT VECTOR NO MATTER WHAT ORDER YOU GIVE IT, HENCE IS LINEARLY DEPENDENT.

3.) MOST TEXTBOOKS GIVE A DIFFERENT DEFINITION OF LINEAR INDEPENDANCE, WHICH WE SHALL PROVE EQUIVALENT TO OURS.

4.) NOTE THAT IF $\{\vec{v}_1, .., \vec{v}_m\}$ FORMS A BASIS OF V, THEN NECESSARILY EACH $\vec{v}_i \in V$

5.) SUPPOSE $V = \text{SPAN}(\vec{v}_1, \ldots, \vec{v}_m)$ BUT THE SET $\{\vec{v}_1, .., \vec{v}_m\}$ IS LINEARLY DEPENDENT. TO OBTAIN A BASIS FOR V, SIMPLY LIST THE SET (IN ANY ORDER) AND DELETE ANY REDUNDANT VECTORS. THE RESULTING SET IS LINEARLY INDEPENDENT, AND STILL SPANS V SINCE ONLY REDUNDANT VECTORS WERE DELETED.

<u>Ex.</u>

IS THE SET

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 6 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 7 \\ 0 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 4 \\ 9 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 8 \\ 7 \\ 6 \end{pmatrix} \right\} \subseteq \mathbb{R}^6$$

LINEARLY INDEPENDENT OR LINEARLY DEPENDENT? ONE CHECKS EASILY THAT IT IS INDEPENDENT.

<u>THEOREM</u>

THE SET $\{\vec{v}_1, \ldots, \vec{v}_m\}$ IS LINEARLY DEPENDENT IFF THERE EXIST $c_1, \ldots, c_m \in \mathbb{R}$ (NOT ALL ZERO) SUCH THAT

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0} .$$

SUCH AN EQUATION IS CALLED A <u>NON-TRIVIAL RELATION</u> AMONGST THE VECTORS $\vec{v}_1, \ldots, \vec{v}_m$.

<u>PROOF</u>:

($\Rightarrow$) SUPPOSE $\{\vec{v}_1, \ldots, \vec{v}_m\}$ IS LINEARLY DEPENDENT. THEN THE LIST $\vec{v}_1, \ldots, \vec{v}_m$ CONTAINS A REDUNDANT VECTOR, SAY $\vec{v}_j$. THUS

$$\vec{v}_j = c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1}$$

FOR SOME $c_1, \ldots, c_{j-1} \in \mathbb{R}$. THEN

$$c_1\vec{v}_1 + \cdots + c_{j-1}\vec{v}_{j-1} + (-1)\vec{v}_j + 0\vec{v}_{j+1} + \cdots + 0\vec{v}_m = \vec{0}$$

IS A NON TRIVIAL RELATION AMONGST $\vec{v}_1, \ldots, \vec{v}_m$ SINCE $-1 \neq 0$.

($\Leftarrow$) LET $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$ BE A NON-TRIVIAL RELATION AMONGST $\vec{v}_1, \ldots, \vec{v}_m$ (i.e. NOT ALL c_i ARE ZERO.) LET j BE THE RIGHTMOST INDEX SUCH THAT $c_j \neq 0$. (i.e. $c_{j+1} = \cdots = c_m = 0$.) THEN

$$\vec{v}_j = \left(\frac{-c_1}{c_j}\right)\vec{v}_1 + \left(\frac{-c_2}{c_j}\right)\vec{v}_2 + \cdots + \left(\frac{-c_{j-1}}{c_j}\right)\vec{v}_{j-1},$$

SHOWING THAT $\vec{v}_j$ IS REDUNDANT IN THE LIST $\vec{v}_1, \ldots, \vec{v}_m$. THEREFORE $\{\vec{v}_1, \ldots, \vec{v}_m\}$ IS LINEARLY INDEPENDENT.

///.

<u>RMK</u>

THIS RESULT ESTABLISHES THAT OUR DEFINITION OF LINEAR DEPENDENCE DOES NOT DEPEND ON THE ORDER OF THE LIST SINCE THE RELATION $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$ IS TRUE (OR FALSE) IRRESPECTIVE OF ANY ORDERING OF THE VECTORS. THUS IF ANY ORDERING OF $\vec{v}_1, \ldots, \vec{v}_m$ YIELDS A REDUNDANT VECTOR, ALL ORDERINGS WILL.

<u>Corollary</u>
The set $\{\vec{v}_1, \dots, \vec{v}_m\} \subseteq \mathbb{R}^n$ is linearly independent iff the equation

$$c_1 \vec{v}_1 + \dots + c_m \vec{v}_m = \vec{0}$$

implies that $c_1 = \dots = c_m = 0$, i.e. there is no non-trivial relation amongst $\vec{v}_1, \dots, \vec{v}_m$.

<u>Proof</u>:
This is just a restatement of the previous theorem.

///

Now let $A = [\vec{v}_1, \dots, \vec{v}_m]$. The preceding corollary can be stated another way:

$\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly independent iff $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$, i.e. iff $\ker(A) = \vec{0}$.

We've seen previously that this condition is equivalent to

$$\operatorname{rank}(A) = m \leq n$$

(Recall summary on p. 23). In particular, we may conclude that a linearly independent set of vectors in $\mathbb{R}^n$ contains at most n vectors.

WE SUMMARIZE THIS IN:

THEOREM
LET $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ AND $A = [\vec{v}_1 \dots \vec{v}_m]$. THEN THE FOLLOWING STATEMENTS ARE EQUIVALENT.

(1) $\{\vec{v}_1 \dots \vec{v}_m\}$ is LINEARLY INDEPENDENT.

(2) NO VECTOR IN THE LIST $\vec{v}_1, \dots, \vec{v}_m$ is REDUNDANT.

(3) NO $\vec{v}_j \in \{\vec{v}_1, \dots, \vec{v}_m\}$ is A LINEAR COMBINATION OF OTHER VECTORS IN THE SET, i.e. $\vec{v}_1, \dots, \vec{v}_{j-1}, \vec{v}_{j+1}, \dots, \vec{v}_m$.

(4) THE EQUATION $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$ IMPLIES THAT $c_1 = \dots = c_m = 0$

(5) $\ker(A) = \{\vec{0}\}$

(6) $\operatorname{rank}(A) = m \leq n$

THE ONLY THING NEW HERE is (3) WHICH WE LEAVE AS AN EXERCISE, e.g. SHOW (4) $\Rightarrow$ (3) AND (3) $\Rightarrow$ (2).

THEOREM

LET $V \subseteq \mathbb{R}^n$ BE A SUBSPACE. THEN $\{\vec{v}_1, \ldots, \vec{v}_m\} \subseteq V$ IS A BASIS OF V IFF EACH $\vec{v} \in V$ CAN BE EXPRESSED UNIQUELY AS A LINEAR COMBINATION

$$(*) \qquad \vec{v} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$$

PROOF

$(\Rightarrow)$ SUPPOSE $\{\vec{v}_1, \ldots, \vec{v}_m\}$ IS A BASIS FOR V AND PICK ANY $\vec{v} \in V$. SINCE $V = \mathrm{SPAN}(\vec{v}_1, \ldots, \vec{v}_m)$ THERE EXIST $c_1, \ldots, c_m \in \mathbb{R}$ SUCH THAT $(*)$ HOLDS. WE MUST SHOW THAT THE COEFFICIENTS c_i $(1 \leq i \leq m)$ ARE UNIQUE. TO THIS END, SUPPOSE ALSO THAT

$$\vec{v} = d_1 \vec{v}_1 + \cdots + d_m \vec{v}_m \quad .$$

SUBTRACTING THE TWO EXPRESSIONS FOR $\vec{v}$ YIELDS

$$(c_1 - d_1)\vec{v}_1 + \ldots + (c_m - d_m)\vec{v}_m = \vec{0} \, .$$

THE LINEAR INDEPENDENCE OF $\{\vec{v}_1, \ldots, \vec{v}_m\}$ IMPLIES $c_1 - d_1 = \ldots = c_m - d_m = 0$, i.e. $c_i = d_i$ $(1 \leq i \leq m)$, PROVING UNIQUENESS.

($\Leftarrow$) NOW SUPPOSE EACH $\vec{v} \in V$ CAN BE EXPRESSED UNIQUELY AS A LINEAR COMBINATION OF $\vec{v}_1, \ldots, \vec{v}_m$ AS IN (*). THIS SAYS IN PARTICULAR THAT

$$V \subseteq \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$$

BUT SINCE V IS A SUBSPACE, AND SINCE $\{\vec{v}_1, \ldots, \vec{v}_m\} \subseteq V$, WE HAVE $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m) \subseteq V$. THUS

$$V = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m).$$

IT REMAINS TO SHOW $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is linearly INDEPENDENT. CONSIDER THE EQUATION

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}.$$

THIS IS A REPRESENTATION OF $\vec{0} \in V$ AS A LINEAR COMBINATION OF $\vec{v}_1, \ldots, \vec{v}_m$, AND AS SUCH, THE COEFFICIENTS c_i $(1 \le i \le m)$ ARE UNIQUE. BUT ALSO

$$0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_m = \vec{0},$$

SO BY THIS UNIQUENESS $c_1 = \cdots = c_m = 0$. THUS $\{\vec{v}_1, \ldots, \vec{v}_m\}$ IS A BASIS OF V, AS REQUIRED.

///

<u>EX.</u> LET

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 \\ 2 & 4 & 8 & 6 \\ 3 & 1 & 7 & -1 \end{pmatrix}$$

FIND A BASIS FOR $\mathrm{im}(A) \subseteq \mathbb{R}^3$ AND A BASIS FOR $\mathrm{KER}(A) \subseteq \mathbb{R}^4$.

LET $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ DENOTE THE COLUMNS OF A. ONE CHECKS EASILY THAT

$$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2 \quad \text{AND} \quad \vec{v}_4 = -\vec{v}_1 + 2\vec{v}_2 ,$$

WHENCE

$$\mathrm{im}(A) = \mathrm{SPAN}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \mathrm{SPAN}(\vec{v}_1, \vec{v}_2) .$$

ONE ALSO CHECKS THAT

$$\mathrm{RREF}\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

WHICH HAS RANK 2, SO $\{\vec{v}_1, \vec{v}_2\}$ is LINEARLY INDEPENDENT.

$\therefore \{\vec{v}_1, \vec{v}_2\}$ is A BASIS OF $\mathrm{im}(A)$.

TO FIND $\mathrm{KER}(A)$ WE NEED TO FIND ALL SOLUTIONS TO THE SYSTEM $A\vec{x} = \vec{0}$, i.e.

$$\left(\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 2 & 4 & 8 & 6 & 0 \\ 3 & 1 & 7 & -1 & 0 \end{array}\right)$$

AFTER PERFORMING GAUSS-JORDAN ELIMINATION WE FIND THE RREF TO BE

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

FROM WHICH WE SEE $\vec{x} \in \mathrm{KER}(A)$ IFF

$$\vec{x} = \begin{pmatrix} -2t+s \\ -t-2s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

WHERE $t, s \in \mathbb{R}$ ARE ARBITRARY. THUS $\mathrm{KER}(A) = \mathrm{SPAN}(\vec{u}_1, \vec{u}_2)$ WHERE

$$\vec{u}_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

ALSO ONE CHECKS

$$\mathrm{RREF}\begin{pmatrix} -2 & 1 \\ -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

SHOWING $\{\vec{u}_1, \vec{u}_2\}$ is LINEARLY INDEPENDENT.

HENCE $\{\vec{u}_1, \vec{u}_2\}$ IS A BASIS OF $\ker(A)$.

LET US ILLUSTRATE THE PREVIOUS THEOREM ON THE SUBSPACE $\operatorname{im}(A) \subseteq \mathbb{R}^3$. PICK FOR INSTANCE

$$\vec{v} = \begin{pmatrix} 3 \\ 14 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \in \operatorname{im}(A).$$

IT MUST BE POSSIBLE TO FIND UNIQUE $c_1, c_2 \in \mathbb{R}$ SUCH THAT $\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2$, i.e.

$$\begin{pmatrix} 3 \\ 14 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$$

TO FIND c_1, c_2 WE SOLVE

$$\begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 14 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 4 & 14 \\ 3 & 1 & 1 \end{array}\right) \longrightarrow \text{GAUSS-JORDAN} \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

THUS $c_1 = -1$, $c_2 = 4$. FURTHERMORE THE RREF OF THIS SYSTEM SHOWS US THAT THE SOLUTION

IS UNIQUE, WHICH IS WHAT THE LAST THEOREM TOLD US TO EXPECT.

EXERCISE

LET $\vec{u} = \begin{pmatrix} -1 \\ -8 \\ 2 \\ 3 \end{pmatrix}$.

CHECK THAT $\vec{u} \in \mathrm{KER}(A)$, AND SHOW THAT THERE EXIST UNIQUE $d_1, d_2 \in \mathbb{R}$ SUCH THAT

$$\vec{u} = d_1 \vec{u}_1 + d_2 \vec{u}_2$$

BY SOLVING A LINEAR SYSTEM.

HW (3.2) 4, 6ab, 8, 10, 14, 16, 18, 22, 24, 26, 48, 50, 52