3.1 IMAGE AND KERNAL

LET $f: A \to B$ BE A FUNCTION. THE IMAGE OF f IS THE SET

$$\text{image}(f) = \{ f(x) \mid x \in A \} \subseteq B$$

NOTE

- WE PREVIOUSLY DEFINED THIS SET AS $\text{range}(f)$, AS DO MANY TEXTS.
- f IS SURJECTIVE IFF $\text{image}(f) = B$.

LET $T: \mathbb{R}^m \to \mathbb{R}^n$ BE LINEAR. WE WRITE $\text{im}(T)$ FOR IMAGE OF T. IF A IS A $n \times m$ MATRIX WE WRITE

$$\text{im}(A) = \text{im}(T_A) \subseteq \mathbb{R}^n.$$

EX. $A = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}$

THEN $\text{im}(A)$ IS THE SET OF ALL VECTORS OF THE FORM

$$A\vec{x} = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 12 \end{pmatrix}$$

WHERE $\vec{x} \in \mathbb{R}^2$ IS ARBITRARY. BUT NOTICE

$$\begin{pmatrix} 4 \\ 12 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

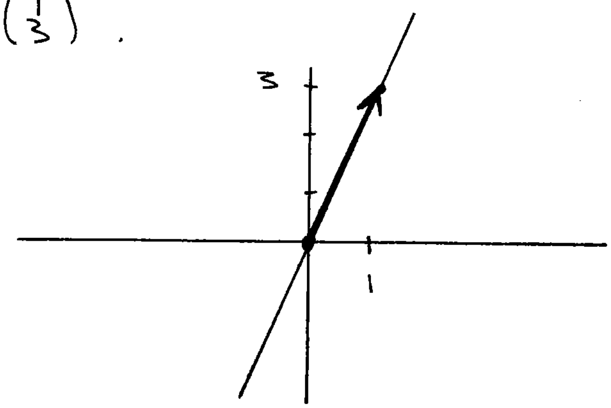
SO THAT

$$A\vec{x} = x_1\begin{pmatrix}1\\3\end{pmatrix} + 4x_2\begin{pmatrix}1\\3\end{pmatrix} = (x_1+4x_2)\begin{pmatrix}1\\3\end{pmatrix}.$$

SINCE x_1, x_2 ARE ARBITRARY WE SEE

$$\operatorname{im}(A) = \left\{ t\begin{pmatrix}1\\3\end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$$

WHICH IS THE LINE THROUGH $\begin{pmatrix}0\\0\end{pmatrix}$ PARALLEL TO $\begin{pmatrix}1\\3\end{pmatrix}$.



NOTE: IN THIS CASE $\operatorname{rank}(A) = 1$.

Ex. $A = \begin{pmatrix}1 & 4\\3 & 11\end{pmatrix}$

OBSERVE $\det(A) = 1\cdot 11 - 3\cdot 4 = -1 \neq 0$. HENCE T_A IS INVERTIBLE, AND THEREFORE BIJECTIVE, WHENCE

$$\operatorname{im}(A) = \mathbb{R}^2$$

NOTICE IN THIS CASE $\operatorname{rank}(A) = 2$.

Ex. $A = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

HERE $\operatorname{rank}(A) = 0$ AND OBVIOUSLY $\operatorname{im}(A) = \{\vec{0}\}$.

MORE GENERALLY LET A BE ANY $n \times m$ MATRIX, SAY WITH COLUMNS

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix} \qquad (\vec{v}_j \in \mathbb{R}^n,\ 1 \le j \le m).$$

THEN $\mathrm{im}(A)$ IS THE SET OF ALL VECTORS OF THE FORM

$$A\vec{x} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$= x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m$$

WHERE $x_1, x_2, \ldots, x_m$ ARE ARBITRARY REAL NUMBERS. RECALL THAT SUCH AN EXPRESSION IS CALLED A LINEAR COMBINATION OF THE VECTORS $\vec{v}_1, \ldots, \vec{v}_m$.

DEFN.
THE SPAN OF A COLLECTION $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ OF VECTORS IS THE SET OF ALL SUCH LINEAR COMBINATIONS

$$\mathrm{SPAN}(\vec{v}_1, \ldots, \vec{v}_m) = \left\{ c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \,\middle|\, c_1, \ldots, c_m \in \mathbb{R} \right\}$$

OBSERVE THAT $\mathrm{SPAN}(\vec{v}_1, \ldots, \vec{v}_m) \subseteq \mathbb{R}^n$.

THUS THE IMAGE OF A LINEAR TRANSFORMATION IS THE SPAN OF THE COLUMNS OF THE CORRESPONDING MATRIX. i.e. IF

$$A = \left[\vec{v}_1 \cdots \vec{v}_m \right]$$

THEN

$$\operatorname{im}(A) = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m).$$

EX.

$$A = \begin{pmatrix} 1 & 2 & 4 & 7 \\ 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \quad 3 \times 4$$

$$\begin{matrix} \| & \| & \| & \| \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{matrix}$$

AND $\operatorname{im}(A) = \operatorname{SPAN}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)$. BUT NOTICE THAT

$$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$$

AND

$$\vec{v}_4 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = 3\vec{v}_1 + 2\vec{v}_2$$

THUS ANY LINEAR COMBINATION OF $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ CAN BE SIMPLIFIED TO A LINEAR COMBINATION OF $\vec{v}_1$ AND $\vec{v}_2$ ONLY.

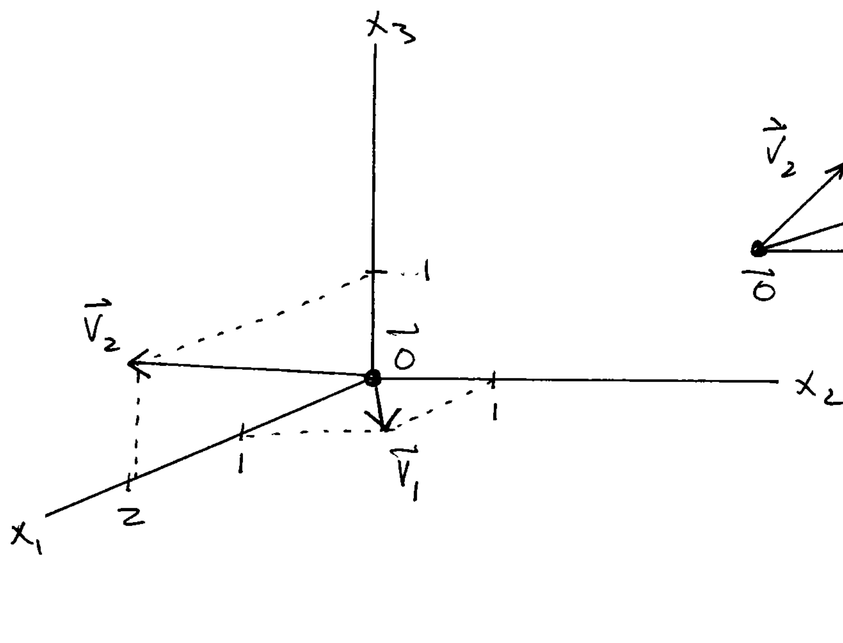
$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$$
$$= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(2\vec{v}_1 + \vec{v}_2) + c_4(3\vec{v}_1 + 2\vec{v}_2)$$
$$= (c_1 + 2c_3 + 3c_4)\vec{v}_1 + (c_2 + c_3 + 2c_4)\vec{v}_2 ,$$

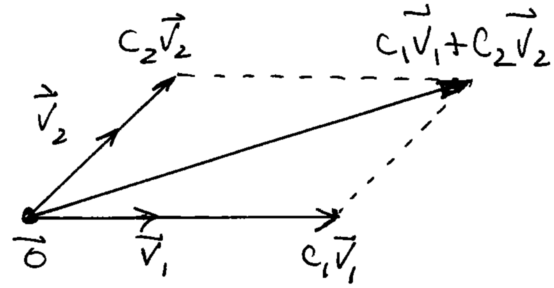
AND WE SEE $\text{SPAN}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \text{SPAN}(\vec{v}_1, \vec{v}_2)$, WHENCE

$$\text{im}(A) = \text{SPAN}(\vec{v}_1, \vec{v}_2)$$

WHICH IS THE PLANE IN $\mathbb{R}^3$ THAT PASSES THROUGH $\vec{0}$ AND CONTAINS THE VECTORS

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{AND} \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$





Properties of im(T)

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be linear. Then

a.) $\vec{0} \in \text{im}(T)$
b.) If $\vec{v}_1, \vec{v}_2 \in \text{im}(T)$, then $\vec{v}_1 + \vec{v}_2 \in \text{im}(T)$
c.) If $\vec{v} \in \text{im}(T)$ and $k \in \mathbb{R}$, then $k\vec{v} \in \text{im}(T)$.

Proof:

a.) $T(\vec{0}) + T(\vec{0}) = T(\vec{0}+\vec{0}) = T(\vec{0})$

$\therefore\ T(\vec{0}) = \vec{0}$

$\therefore\ \vec{0} \in \text{im}(T)$

b.) Let $\vec{v}_1, \vec{v}_2 \in \text{im}(T)$. Then there exist $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^m$ such that

$$T(\vec{u}_1) = \vec{v}_1 \quad \text{and} \quad T(\vec{u}_2) = \vec{v}_2$$

$$\therefore\ \vec{v}_1 + \vec{v}_2 = T(\vec{u}_1) + T(\vec{u}_2) = T(\vec{u}_1 + \vec{u}_2) \in \text{im}(T).$$

c.) Let $\vec{v} \in \text{im}(T)$ and $k \in \mathbb{R}$. Then there exists $\vec{u} \in \mathbb{R}^m$ such that $T(\vec{u}) = \vec{v}$.

$$\therefore\ k\vec{v} = kT(\vec{u}) = T(k\vec{u}) \in \text{im}(T).$$

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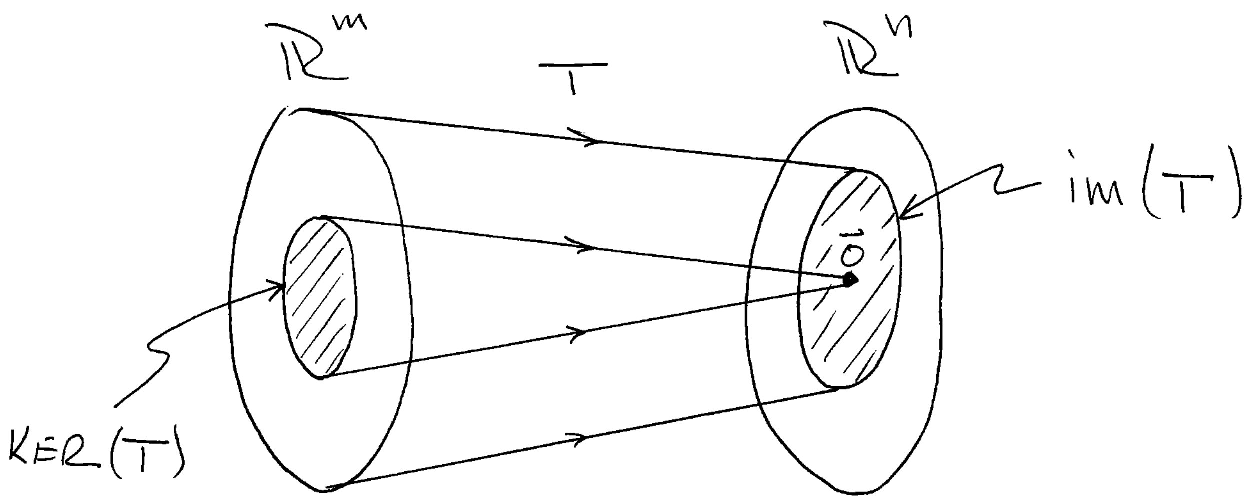
We are often interested in the zeros of a function, i.e. the set of all elements in the domain which are mapped to zero in the codomain.

DEFN
LET $T : \mathbb{R}^m \to \mathbb{R}^n$ BE linear. THE KERNAL OF T is THE SET OF All $\vec{x} \in \mathbb{R}^m$ which ARE MAPPED TO $\vec{0} \in \mathbb{R}^n$, i.e.

$$\mathrm{KER}(T) = \{ \vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0} \}$$

IF A is AN $n \times m$ MATRIX WE WRITE $\mathrm{KER}(A)$ FOR $\mathrm{KER}(T_A)$.



NOTE
- $\mathrm{im}(T) \subseteq \mathbb{R}^n$
- $\mathrm{KER}(T) \subseteq \mathbb{R}^m$.

EX. LET $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \end{pmatrix}$, FIND $\mathrm{KER}(A) \subseteq \mathbb{R}^3$.

WE MUST FIND All solutions TO THE linear SYSTEM $A\vec{x} = \vec{0}$, i.e.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 2 & 5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right)$$

$x_3 = t$ IS FREE

$$\therefore \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -3t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

$\therefore$ $\text{KER}(A) = \text{SPAN}\left(\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right) \subseteq \mathbb{R}^3$, which is a line in $\mathbb{R}^3$.

<u>PROPERTIES OF KER(T)</u>

LET $T: \mathbb{R}^m \to \mathbb{R}^n$ BE LINEAR. THEN

a.) $\vec{0} \in \text{KER}(T)$
b.) IF $\vec{u}_1, \vec{u}_2 \in \text{KER}(T)$, THEN $\vec{u}_1 + \vec{u}_2 \in \text{KER}(T)$.
c.) IF $\vec{u} \in \text{KER}(T)$ AND $k \in \mathbb{R}$, THEN $k\vec{u} \in \text{KER}(T)$.

<u>PROOF</u>: EXERCISE

THEOREM

LET $T:\mathbb{R}^m \to \mathbb{R}^n$ BE LINEAR AND LET A BE THE CORRESPONDING $n \times m$ MATRIX. THEN

(1) $\ker(T) = \{\vec{0}\}$ IFF T IS INJECTIVE
(2) $\ker(A) = \{\vec{0}\}$ IFF $\operatorname{rank}(A) = m \leq n$.

ALSO, IF $m = n$ WE HAVE:

(3) $\ker(A) = \{\vec{0}\}$ IFF A IS INVERTIBLE
(4) T IS INJECTIVE IFF T IS SURJECTIVE.

PROOF:

(1)

($\Rightarrow$) SUPPOSE $\ker(T) = \{\vec{0}\}$. WE MUST SHOW THAT $T(\vec{x}) = T(\vec{y}) \Rightarrow \vec{x} = \vec{y}$, FOR ANY VECTORS $\vec{x}, \vec{y} \in \mathbb{R}^m$, SHOWING THAT T IS INJECTIVE.

$$\begin{aligned}
T(\vec{x}) = T(\vec{y}) &\Rightarrow T(\vec{x}) - T(\vec{y}) = \vec{0} \\
&\Rightarrow T(\vec{x} - \vec{y}) = \vec{0} \\
&\Rightarrow \vec{x} - \vec{y} \in \ker(T) \\
&\Rightarrow \vec{x} - \vec{y} = \vec{0} \quad (\text{SINCE } \ker(T) = \{\vec{0}\}) \\
&\Rightarrow \vec{x} = \vec{y}.
\end{aligned}$$

($\Leftarrow$) NOW SUPPOSE T IS INJECTIVE. PICK ANY $\vec{x} \in \ker(T)$, SO THAT $T(\vec{x}) = \vec{0}$. BUT ALSO $T(\vec{0}) = \vec{0}$. THUS $T(\vec{x}) = T(\vec{0})$. SINCE T IS INJECTIVE WE HAVE $\vec{x} = \vec{0}$. SINCE $\vec{x}$ WAS ARBITRARY, WE CONCLUDE

$$\ker(T) = \{\vec{0}\}.$$

(2)

OBSERVE THAT $\ker(A) = \{\vec{0}\}$ MEANS PRECISELY THAT $A\vec{x} = \vec{0}$ HAS ONLY THE SOLUTION $\vec{x} = \vec{0}$. RECALL THE SUMMARY FROM PAGE 23 OF THE NOTES:

	CASE	# SOLNS TO $(A \mid \vec{b})$
1	$r = m = n$	1
2	$r = n < m$	∞
3	$r = m < n$	0 OR 1
4	$r < m = n$	
5	$r < m < n$	0 OR ∞
6	$r < n < m$	

HERE $r = \text{rank}(A)$. NOTE THAT $A\vec{x} = \vec{0}$ ALWAYS AT LEAST ONE SOLUTION, NAMELY $\vec{x} = \vec{0}$. THUS WHEN $\vec{b} = \vec{0}$ THE POSSIBILITY OF NO SOLUTIONS GOES AWAY IN CASES 3 – 6.

WE SEE FROM THE ABOVE CHART THAT $A\vec{x} = \vec{0}$ HAS A UNIQUE SOLUTION IFF EITHER CASE 1 OR CASE 3 HOLD, WHICH IS TRUE IFF

$$r = m \le n$$

AS REQUIRED.

(3)

SUPPOSE $m=n$. BY THE SAME SUMMARY $A\vec{x}=\vec{0}$ HAS A UNIQUE SOLUTION IFF CASE 1 HOLDS, WHICH SAYS $r=n$, AND HENCE $RREF(A)=I_n$. BUT AS WE'VE SEEN PREVIOUSLY (p. 54-55 OF NOTES), THIS IS EQUIVALENT TO A BEING INVERTIBLE. THUS $KER(A)=\{\vec{0}\}$ IFF A IS INVERTIBLE.

(4)

SUPPOSE $m=n$. THEN
($\Rightarrow$) IF T IS INJECTIVE THEN $KER(A)=KER(T)=\{\vec{0}\}$, BY (1). THEREFORE A (AND HENCE T) IS INVERTIBLE, BY (3). WE CONCLUDE T IS SURJECTIVE.

($\Leftarrow$) IF T IS SURJECTIVE THEN THE LINEAR SYSTEM $A\vec{x}=\vec{b}$ IS CONSISTENT FOR ALL $\vec{b}\in\mathbb{R}^n$. THEREFORE CASES 3-6 OF THE SUMMARY ARE IMPOSSIBLE. CASE 2 CONTRADICTS $m=n$, SO ONLY CASE 1 REMAINS AS A POSSIBILITY. THEREFORE $r=n$, WHENCE T IS INVERTIBLE, AND HENCE T IS INJECTIVE

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HW (3.1): 2-16 even, 24, 30, 32, 34, 44ab, 48ab