2.4 MATRIX PRODUCTS

LET A, B, C BE SETS AND SUPPOSE

$$f : A \to B \quad \text{AND} \quad g : B \to C$$

THE COMPOSITION $g \circ f : A \to C$ IS THE FUNCTION GIVEN BY

$$(g \circ f)(x) = g(f(x)) \quad \text{FOR ALL } x \in A$$

EX. $A = B = C = \mathbb{R}$, $f(x) = x+5$, $g(x) = x^2$

$$g \circ f(x) = g(f(x)) = g(x+5) = (x+5)^2$$
$$f \circ g(x) = f(g(x)) = f(x^2) = x^2 + 5$$

IN GENERAL $g \circ f \neq f \circ g$, EVEN WHEN BOTH COMPOSITIONS ARE DEFINED.

LEMMA

(1) f and g injective $\Rightarrow$ $g \circ f$ INJECTIVE
(2) $g \circ f$ INJECTIVE $\Rightarrow$ f INJECTIVE
(3) f AND g SURJECTIVE $\Rightarrow$ $g \circ f$ SURJECTIVE
(4) $g \circ f$ SURJECTIVE $\Rightarrow$ g SURJECTIVE

PROOF:

(1) LET $x_1, x_2 \in A$ AND SUPPOSE $g \circ f(x_1) = g \circ f(x_2)$. WE MUST SHOW $x_1 = x_2$. WE HAVE

$$g(f(x_1)) = g(f(x_2))$$

$\therefore \quad f(x_1) = f(x_2)$ SINCE g IS INJECTIVE.

$\therefore \quad x_1 = x_2$ SINCE f IS INJECTIVE.

(2) EXERCISE

(3) EXERCISE

(4) LET $z \in C$. WE MUST SHOW $g(y) = z$ FOR SOME $y \in B$. SINCE $g \circ f$ IS SURJECTIVE, WE KNOW $g \circ f(x) = z$ FOR SOME $x \in A$. LET $y = f(x)$. THEN $g(y) = g(f(x)) = g \circ f(x) = z$, AS REQUIRED.

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<u>THEOREM</u>

IF $f : A \to B$ AND $g : B \to C$ ARE BOTH INVERTIBLE, THEN SO IS THE COMPOSITION $g \circ f : A \to C$. FURTHERMORE

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

<u>PROOF</u>:

SINCE f AND g ARE INVERTIBLE, BOTH ARE BIJECTIVE, HENCE $g \circ f$ IS BIJECTIVE BY THE LEMMA AND THEREFORE $g \circ f$ IS INVERTIBLE.

LET $x \in A$. THEN

$$(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) = f^{-1}(g^{-1}(g(f(x))))$$
$$= f^{-1}(f(x))$$
$$= x \,.$$

LIKEWISE, IF $z \in C$, THEN

$$(g \circ f) \circ (f^{-1} \circ g^{-1})(z) = g(f(f^{-1}(g^{-1}(z))))$$
$$= g(g^{-1}(z))$$
$$= z \,.$$

$\therefore \ (g \circ f)^{-1} = f^{-1} \circ g^{-1}$, AS CLAIMED. ///

NOW SUPPOSE A IS AN $n \times p$ MATRIX AND B IS A $p \times m$ MATRIX. LET

$$T_B : \mathbb{R}^m \to \mathbb{R}^p \quad , \quad T_A : \mathbb{R}^p \to \mathbb{R}^n$$

BE THE CORRESPONDING LINEAR TRANSFORMATIONS.

OBSERVE THAT $T_A \circ T_B : \mathbb{R}^m \to \mathbb{R}^n$ IS ALSO LINEAR, FOR IF $\vec{x}, \vec{y} \in \mathbb{R}^m$ AND $k \in \mathbb{R}$

$$(1) \ T_A \circ T_B(\vec{x} + \vec{y}) = T_A(T_B(\vec{x} + \vec{y}))$$
$$= T_A(T_B(\vec{x}) + T_B(\vec{y}))$$
$$= T_A(T_B(\vec{x})) + T_A(T_B(\vec{y}))$$
$$= T_A \circ T_B(\vec{x}) + T_A \circ T_B(\vec{y}) \,,$$

AND

$$(2)\ T_A \circ T_B(k\vec{x}) = T_A(T_B(k\vec{x}))$$
$$= T_A(k\,T_B(\vec{x}))$$
$$= k\,T_A(T_B(\vec{x}))$$
$$= k(T_A \circ T_B(x)).$$

THUS WE MAY ASK WHAT MATRIX CORRESPONDS TO $T_A \circ T_B$, AS USUAL WE NEED ONLY DETERMINE ITS COLUMNS:

$$\left[T_A \circ T_B(\vec{e}_1) \cdots T_A \circ T_B(\vec{e}_j) \cdots T_A \circ T_B(\vec{e}_m) \right]$$

WHERE $\vec{e}_1 \cdots \vec{e}_j \cdots \vec{e}_m$ IS THE STANDARD BASIS IN $\mathbb{R}^m$.

SUPPOSE B HAS COLUMNS

$$B = \left[\vec{w}_1 \ \cdots \ \vec{w}_j \ \cdots \ \vec{w}_m \right]$$

THEN $\vec{w}_j = T_B(\vec{e}_j)$ FOR $1 \le j \le m$, AND THE j^{TH} COLUMN OF THE MATRIX FOR $T_A \circ T_B$ IS

$$T_A \circ T_B(\vec{e}_j) = T_A(T_B(\vec{e}_j)) = T_A(\vec{w}_j) = A\vec{w}_j.$$

FOR $1 \le j \le m$.

THUS THE MATRIX FOR $T_A \circ T_B$ HAS COLUMNS

$$\left[A\vec{w}_1 \cdots A\vec{w}_j \cdots A\vec{w}_m \right].$$

DEFN

SUPPOSE

$$A = \begin{array}{cl} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} & \begin{array}{l} \\ \\ i^{TH} \text{ ROW} \\ \\ (n \times p) \end{array} \end{array}$$

AND

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1m} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2m} \\ \vdots & & \vdots & & \vdots \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pm} \end{pmatrix} \quad (p \times m)$$

j^{TH} COLUMN

WE DEFINE THE PRODUCT AB TO BE THE $n \times m$ MATRIX WHOSE i^{TH} ROW, j^{TH} COLUMN CONTAINS

$$a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

FOR $1 \le i \le n$, $1 \le j \le m$.

THUS IF $B = [\vec{w}_1 \cdots \vec{w}_j \cdots \vec{w}_m]$, SO THAT

$$\vec{w}_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix} \quad (p\times 1)$$

THEN

$$A\vec{w}_j = \begin{pmatrix} a_{11} b_{1j} + a_{12} b_{2j} + \cdots + a_{1p} b_{pj} \\ \vdots \\ a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj} \\ \vdots \\ a_{n1} b_{1j} + a_{n2} b_{2j} + \cdots + a_{np} b_{pj} \end{pmatrix} \quad (n\times 1)$$

AND THEREFORE

$$AB = [A\vec{w}_1 \cdots A\vec{w}_j \cdots A\vec{w}_m] \quad (n\times m)$$

WHICH IS PRECISELY THE MATRIX FOR $T_A \circ T_B$.

<u>THEOREM</u> $T_A \circ T_B = T_{AB}$

<u>EX.</u> $n=2,\ p=3,\ m=4$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & -1 & 0 & 5 \\ 1 & 1 & 3 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 3 & 5 & 11 & 8 \\ 2 & 0 & 11 & 22 \end{pmatrix}$$

Ex. $n=p=m=2$

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

ONE CHECKS EASILY THAT IN GENERAL

- $AB \neq BA$

BUT
- $\det(AB)=\det(A)\cdot\det(B)=\det(BA)$.

WE WILL USE FREQUENTLY THAT :

$$T_A = T_B \iff A = B$$

PROOF:

($\Leftarrow$) DEFN. OF T_A, T_B

($\Rightarrow$) $T_A = T_B \Rightarrow T_A(\vec{x}) = T_B(\vec{x})$ FOR ALL $\vec{x} \in \mathbb{R}^m$

$\Rightarrow A\vec{x} = B\vec{x}$ FOR ALL $\vec{x} \in \mathbb{R}^m$

$$\Rightarrow \begin{cases} A\vec{e}_1 = B\vec{e}_1 \\ \vdots \\ A\vec{e}_m = B\vec{e}_m \end{cases}$$

$\Rightarrow$ COLUMNS OF A ARE IDENTICAL TO COLUMNS OF B

$\Rightarrow A = B$.

///.

FOR ANY $\vec{x} \in \mathbb{R}^n$ WE HAVE

$$T_{I_n}(\vec{x}) = I_n \vec{x} = \vec{x}$$

SO $T_{I_n} : \mathbb{R}^n \to \mathbb{R}^n$ IS THE IDENTITY MAPPING.

LET A BE AN $n \times n$ INVERTIBLE MATRIX, SO THAT $T_A^{-1} = T_{A^{-1}}$. THEN

$$T_{AA^{-1}} = T_A \circ T_{A^{-1}} = T_A \circ T_A^{-1} = T_{I_n}$$

AND

$$T_{A^{-1}A} = T_{A^{-1}} \circ T_A = T_A^{-1} \circ T_A = T_{I_n}$$

WHENCE $AA^{-1} = I_n = A^{-1}A$.

<u>EX.</u>

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1}A = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

AS EXPECTED.

LET A BE AN $n \times m$ MATRIX SO $T_A : \mathbb{R}^m \to \mathbb{R}^n$. THEN

$$T_{AI_m} = T_A \circ T_{I_m} = T_A$$

AND

$$T_{I_nA} = T_{I_n} \circ T_A = T_A$$

THUS $I_nA = A = AI_m$.

<u>EXERCISE</u>

LET f, g, h BE FUNCTIONS. SHOW THAT

$$f \circ (g \circ h) = (f \circ g) \circ h$$

PROVIDED THE COMPOSITIONS ARE DEFINED.

LET A $(n \times p)$, B $(p \times q)$, AND C $(q \times m)$ BE MATRICES, SO THAT

$$T_C : \mathbb{R}^m \to \mathbb{R}^q, \; T_B : \mathbb{R}^q \to \mathbb{R}^p, \; T_A : \mathbb{R}^p \to \mathbb{R}^n.$$

THEN

$$T_{A(BC)} = T_A \circ T_{BC} = T_A \circ (T_B \circ T_C)$$
$$= (T_A \circ T_B) \circ T_C = T_{AB} \circ T_C = T_{(AB)C},$$

WHENCE $A(BC) = (AB)C$, i.e. MATRIX MULTIPLICATION IS ASSOCIATIVE.

LET A, B BE INVERTIBLE $n \times n$ MATRICES, SO T_A, T_B ARE INVERTIBLE LINEAR MAPS, AND HENCE $T_A \circ T_B = T_{AB}$ IS INVERTIBLE, $\therefore$ AB IS AN INVERTIBLE $n \times n$ MATRIX AND

$$T_{(AB)^{-1}} = T_{AB}^{-1} = (T_A \circ T_B)^{-1} = T_B^{-1} \circ T_A^{-1}$$

$$= T_{B^{-1}} \circ T_{A^{-1}} = T_{B^{-1}A^{-1}}$$

THEREFORE $(AB)^{-1} = B^{-1}A^{-1}$.

EXERCISE

LET A, B BE $n \times p$ MATRICES, AND C, D BE $p \times m$ MATRICES. LET $k \in \mathbb{R}$. SHOW THAT

(1) $A(C+D) = AC + AD$
(2) $(A+B)C = AC + BC$
(3) $k(AC) = (kA)C = A(kC)$

PROPERTIES (1) & (2) SAY THAT MATRIX MULTIPLICATION DISTRIBUTES OVER MATRIX ADDITION.

PARTITIONED MATRICES

IT IS SOMETIMES USEFUL TO PARTITION MATRICES INTO BLOCKS THEN OPERATE ON THE SUBMATRICES.

EX.

$$\left(\begin{array}{c|ccc} 2 & 1 & 7 & 3 \\ \hline 0 & 5 & -1 & 4 \\ 6 & 2 & 1 & -1 \end{array}\right)\left(\begin{array}{cc} 1 & 1 \\ \hline 0 & -1 \\ -2 & 3 \\ 4 & 0 \end{array}\right) = \left(\begin{array}{cc} 0 & 22 \\ \hline 18 & -8 \\ 0 & 7 \end{array}\right)$$

$$3\times 4 \qquad\qquad 4\times 2 \qquad\qquad 3\times 2$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{pmatrix}$$

$$2\times 2 \qquad\qquad 2\times 1 \qquad\qquad 2\times 1$$

$$A_{11} = (2) \quad A_{12} = (1\ 7\ 3) \qquad B_1 = (1\ 1)$$

$$A_{21} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 5 & -1 & 4 \\ 2 & 1 & -1 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0 & -1 \\ -2 & 3 \\ 4 & 0 \end{pmatrix}$$

$$A_{11}B_1 = (2\ \ 2) \quad A_{12}B_2 = (-2\ \ 20)$$

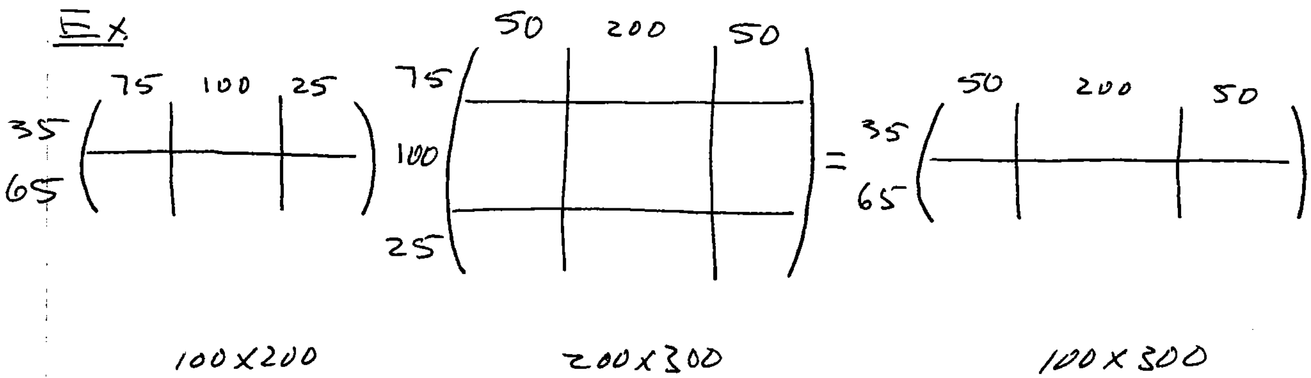
$$A_{21}B_1 = \begin{pmatrix} 0 & 0 \\ 6 & 6 \end{pmatrix} \quad A_{22}B_2 = \begin{pmatrix} 18 & -8 \\ -6 & 1 \end{pmatrix}$$

THERE ARE MANY WAYS TO PARTITION A MATRIX PRODUCT SUCH AS

$$A \cdot B = C$$

THE ROWS OF A AND COLUMNS OF B CAN BE PARTITIONED IN ANY WAY. THE PRODUCT C THEN INHERITS THOSE PARTITIONS ON ITS ROWS AND COLUMNS. HOWEVER THE PARTITION OF THE COLUMNS OF A MUST MATCH THE PARTITION OF THE ROWS OF B.

<u>EX</u>



<u>SOME FINE POINTS ON INVERTIBILITY</u>

<u>THEOREM</u>

LET A BE AN $n \times n$ MATRIX. THEN THE FOLLOWING ARE EQUIVALENT.

(1) A is INVERTIBLE

(2) $\text{rank}(A) = n$

(3) $\text{RREF}(A) = I_n$

(4) $A\vec{x} = \vec{b}$ HAS A UNIQUE SOLUTION FOR ALL $\vec{b} \in \mathbb{R}^n$.

(5) THERE EXISTS AN MATRIX B SUCH THAT $AB = BA = I_n$.

ALL THIS HAS BEEN PROVED, WITH (5) BEING THE MOST RECENT ADDITION. HOWEVER, WE CAN REPLACE (4) AND (5) WITH THE SLIGHTLY WEAKER.

(4') $A\vec{x}=\vec{0}$ HAS THE UNIQUE SOLUTION $\vec{x}=\vec{0}$
(5') THERE EXISTS AN $n \times n$ MATRIX B SUCH THAT $BA = I_n$.

PROOF:
OBVIOUSLY $(4) \Rightarrow (4')$. BUT ALSO $(4') \Rightarrow 2$ BY THE SUMMARY ON P. 23 (§ 1.54) OF THE NOTES, WHICH SAYS $\text{rank}(A) < m = n \Rightarrow$ 0 OR ∞ SOLUTIONS. $\therefore$ $(4) \Leftrightarrow (4')$.

IT IS ALSO OBVIOUS THAT $(5) \Rightarrow (5')$. WE NOW SHOW $(5') \Rightarrow (4')$ WHENCE $(5) \Leftrightarrow (5')$. IF $A\vec{x}=\vec{0}$ THEN

$$\vec{x} = I\vec{x} = (BA)\vec{x} = B(A\vec{x}) = B\vec{0} = 0$$

USING (5'), WHENCE $A\vec{x}=\vec{0}$ HAS AS IT'S ONLY SOLUTION $\vec{x}=\vec{0}$.

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HW (2.4) 2-18 even, 26, 28, 36, 38, 40, 44, 64