2.3 INVERTIBILITY

WHEN IS A LINEAR TRANSFORMATION $T: \mathbb{R}^m \to \mathbb{R}^n$ INVERTIBLE? FIRST OBSERVE THAT IF T IS INVERTIBLE, THEN ITS INVERSE IS ALSO A LINEAR TRANSFORMATION.

TO SEE THIS SUPPOSE $L: \mathbb{R}^n \to \mathbb{R}^m$ IS THE INVERSE OF A LINEAR TRANSFORMATION $T: \mathbb{R}^m \to \mathbb{R}^n$. THUS

$$\vec{u} = T(\vec{x}) \quad \text{IFF} \quad \vec{x} = L(\vec{u})$$

i.e.

$$L(T(\vec{x})) = \vec{x} \quad \text{FOR ALL } \vec{x} \in \mathbb{R}^m$$

and $$T(L(\vec{u})) = \vec{u} \quad \text{FOR ALL } \vec{u} \in \mathbb{R}^n$$

NOW LET $\vec{u}, \vec{v} \in \mathbb{R}^n$. SINCE T IS BIJECTIVE (BEING INVERTIBLE), THERE EXIST UNIQUE $\vec{x}, \vec{y} \in \mathbb{R}^m$ SUCH THAT

$$T(\vec{x}) = \vec{u} \quad \text{and} \quad T(\vec{y}) = \vec{v}$$

SO ALSO

$$L(\vec{u}) = \vec{x} \quad \text{and} \quad L(\vec{v}) = \vec{y}.$$

THEN
$$\begin{aligned} L(\vec{u}+\vec{v}) &= L(T(\vec{x}) + T(\vec{y})) \\ &= L(T(\vec{x}+\vec{y})) \\ &= \vec{x}+\vec{y} \\ &= L(\vec{u}) + L(\vec{v}) \end{aligned}$$

NOW LET $k \in \mathbb{R}$. THEN

$$\begin{aligned} L(k\vec{u}) &= L(kT(\vec{x})) \\ &= L(T(k\vec{x})) \\ &= k\vec{x} \\ &= kL(\vec{u}), \end{aligned}$$

WHENCE L IS A LINEAR TRANSFORMATION AS CLAIMED.

IF A IS THE MATRIX FOR $T : \mathbb{R}^m \to \mathbb{R}^n$, WE WILL DENOTE BY A^{-1} THE MATRIX FOR $T^{-1} : \mathbb{R}^n \to \mathbb{R}^m$.

NOW IF T IS INVERTIBLE, THEN THE LINEAR SYSTEM OF EQUATIONS

$$(*) \qquad A\vec{x} = \vec{b}$$

HAS EXACTLY ONE SOLUTION FOR EVERY $\vec{b} \in \mathbb{R}^n$. WHY? IT HAS

- AT LEAST ONE SOLUTION SINCE T IS SURJECTIVE
- AT MOST ONE SOLUTION SINCE T IS INJECTIVE

THAT UNIQUE SOLUTION IS $\vec{x} = A^{-1}\vec{b}$.

LET $r = \text{rank}(A)$ AND RECALL THE SUMMARY ON P. 23 OF THESE NOTES:

	# solns
1.) $r = m = n$	1
2.) $r = n < m$	∞
3.) $r = m < n$	0, 1
4.) $r < m = n$	$0, \infty$
5.) $r < m < n$	$0, \infty$
6.) $r < n < m$	$0, \infty$

THE ONLY CASES WHICH ADMIT A UNIQUE SOLUTION ARE (1) AND (3). BUT EQUATION (*) WAS TO HAVE A UNIQUE SOLUTION FOR EVERY $\vec{b} \in \mathbb{R}^n$. THIS MAKES CASE (3) IMPOSSIBLE, FOR IF $r = m < n$ THERE IS ALWAYS SOME CHOICE OF $\vec{b}$ WHICH GIVES NO SOLUTION.

Ex. $r = m = 2,\ n = 3$

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

THUS IF $T : \mathbb{R}^m \to \mathbb{R}^n$ IS AN INVERTIBLE LINEAR TRANSFORMATION WITH MATRIX A, THEN $m = n = \text{rank}(A)$.

ON THE OTHER HAND, SUPPOSE CASE (i) HOLDS. THEN (*) HAS A UNIQUE SOLUTION FOR ALL $\vec{b} \in \mathbb{R}^n$. THIS SAYS PRECISELY THAT $T: \mathbb{R}^n \to \mathbb{R}^n$ IS BIJECTIVE, AND HENCE IS INVERTIBLE. WE HAVE PROVED

THEOREM

A LINEAR TRANSFORMATION $T: \mathbb{R}^m \to \mathbb{R}^n$ WITH MATRIX A IS INVERTIBLE IFF $m = n = \text{rank}(A)$.

IN THIS CASE

$$RREF(A) = I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

HOW CAN WE FIND A^{-1}? IT IS SUFFICIENT TO DETERMINE THE VECTORS

$$A^{-1}\vec{e}_1,\ A^{-1}\vec{e}_2,\ \ldots,\ A^{-1}\vec{e}_n$$

WHERE $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ IS THE STANDARD BASIS IN $\mathbb{R}^n$, SINCE THESE ARE PRECISELY THE COLUMNS OF A^{-1}.

i.e.

$$A^{-1} = \left[A^{-1}\vec{e}_1 \;\; A^{-1}\vec{e}_2 \;\; \cdots \;\; A^{-1}\vec{e}_n \right]$$

THUS WE MUST SOLVE n LINEAR SYSTEMS OF EQUATIONS:

$$\begin{aligned}
A\vec{x} &= \vec{e}_1 \xrightarrow{\text{GAUSS-JORDAN}} \vec{x} = A^{-1}\vec{e}_1 \\
A\vec{x} &= \vec{e}_2 \longrightarrow \vec{x} = A^{-1}\vec{e}_2 \\
&\vdots \\
A\vec{x} &= \vec{e}_n \longrightarrow \vec{x} = A^{-1}\vec{e}_n .
\end{aligned}$$

BUT OBSERVE THAT THE SAME SET OF ELEMENTARY ROW OPERATIONS SOLVE EACH OF THESE SYSTEMS. THAT SET CONSISTS OF EXACTLY THOSE EROs WHICH TRANSFORM A INTO $RREF(A) = I_n$.

THUS WE CAN SOLVE ALL n SYSTEMS SIMULTANEOUSLY BY PERFORMING GAUSS-JORDAN ELIMINATION ON THE $n \times (2n)$ AUGMENTED MATRIX

$$(A \mid I_n)$$

TO OBTAIN

$$RREF(A \mid I_n) = (I_n \mid B)$$

WHEN THIS IS DONE THEN $B = A^{-1}$. SINCE ITS COLUMNS ARE THE SOLUTION VECTORS TO THE ABOVE n LINEAR SYSTEMS, i.e.

$$B = \left[A^{-1}\vec{e}_1 \ A^{-1}\vec{e}_2 \cdots A^{-1}\vec{e}_n \right] = A^{-1} .$$

<u>Ex.</u> $n=3$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 4 & 3 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ -4 \cdot \mathrm{I} \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & -5 & -12 & -4 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \div(-1) \\ \\ \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & -5 & -12 & -4 & 0 & 1 \end{array}\right) \begin{array}{l} -2 \cdot \mathrm{II} \\ \\ +5 \cdot \mathrm{II} \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 1 & 2 & 0 \\ 0 & 1 & -12 & 0 & -1 & 0 \\ 0 & 0 & -22 & -4 & -5 & 1 \end{array}\right) \begin{array}{l} \\ \\ \div(-22) \end{array}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{2}{11} & \frac{5}{22} & \frac{-1}{22} \end{array}\right) \begin{array}{l} -7 \cdot \mathrm{III} \\ +2 \cdot \mathrm{III} \\ \\ \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-3}{11} & \frac{9}{22} & \frac{7}{22} \\ 0 & 1 & 0 & \frac{4}{11} & \frac{-6}{11} & \frac{-1}{11} \\ 0 & 0 & 1 & \frac{2}{11} & \frac{5}{22} & \frac{-1}{22} \end{array}\right)$$

THUS

$$A^{-1} = \begin{pmatrix} \frac{-3}{11} & \frac{9}{22} & \frac{7}{22} \\ \frac{4}{11} & \frac{-6}{11} & \frac{-1}{11} \\ \frac{2}{11} & \frac{5}{22} & \frac{-1}{22} \end{pmatrix}$$

$$= \frac{1}{22} \begin{pmatrix} -6 & 9 & 7 \\ 8 & -12 & -2 \\ 4 & 5 & -1 \end{pmatrix}$$

<u>Ex.</u> $n = 2$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \div a \qquad (\text{ASSUME } a \neq 0)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right) - c \cdot \mathrm{I}$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right) \div \left(\frac{ad-bc}{a}\right) \qquad \left(\begin{array}{c}\text{assume} \\ ad-bc \neq 0\end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) - \frac{b}{a} \cdot \mathrm{II}$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$\therefore \quad A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

OUR FIRST STEP REQUIRES $a \neq 0$. NOW SUPPOSE $a = 0$, BUT STILL $ad-bc \neq 0$. THEN NECESSARILY $b \neq 0$ AND $c \neq 0$. IN THIS CASE

$$\left(\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right) \begin{array}{l} \div c \\ \div b \end{array}$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right) \begin{array}{l} -\frac{d}{c}\cdot \mathrm{II} \\ \end{array}$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{-d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right)$$

$$\therefore \quad A^{-1} = \begin{pmatrix} \frac{-d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{pmatrix} = \frac{1}{-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

WE SEE THAT IN EITHER CASE, IF $ad-bc \neq 0$, THEN A IS INVERTIBLE AND

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

THE QUANTITY $ad-bc$ IS CALLED THE <u>DETERMINANT</u> OF A:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

THE ELEMENTARY ROW OPERATIONS HAVE A PREDICTABLE EFFECT ON THE DETERMINANT OF A 2×2 MATRIX

<u>ERO 1</u>: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ <u>det</u>: $bc-ad = -(ad-bc)$

<u>ERO 2</u>: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} ka & kb \\ c & d \end{pmatrix}$ $kad-kbc = k(ad-bc)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ kc & kd \end{pmatrix}$ $kad-kbc = k(ad-bc)$

<u>ERO 3</u>: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c+ka & d+kb \end{pmatrix}$ $a(d+kb)-b(c+ka) = ad-bc$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix}$ $(a+kc)d-(b+kd)c = ad-bc$

IN PARTICULAR WE SEE THAT IF $\det(A) \neq 0$ (OR $= 0$) BEFORE AN ERO IS PERFORMED, THEN THE RESULTING MATRIX RETAINS THAT SAME PROPERTY.

OBSERVE THAT IF A IS ANY 2×2 MATRIX, THERE ARE EXACTLY 4 POSSIBILITIES FOR ITS RREF.

RREF(A)	rank(A)	INVERTIBLE?	det(RREF(A))
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	YES	1
$\begin{pmatrix} 1 & * \\ 0 & 0 \end{pmatrix}$	1	NO	0
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	1	NO	0
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0	NO	0

THUS A IS INVERTIBLE IFF $\det(\text{RREF}(A)) \neq 0$. BUT AS WE'VE SEEN THE ERO'S ARE REVERSIBLE, AND THEY PRESERVE THE PROPERTY OF HAVING NON-ZERO DETERMINANT. THUS $\det(\text{RREF}(A)) \neq 0$ IFF $\det(A) \neq 0$, PROVING:

<u>THEOREM</u>

A 2×2 MATRIX $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ IS INVERTIBLE IFF $\det(A) = ad - bc \neq 0$, IN WHICH CASE

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

HW (2.3) 2, 4, 10, 12, 14, 20, 22, 24, 26, 32, 34ab, 38, 40, 48, 54.