2.1 LINEAR TRANSFORMATIONS

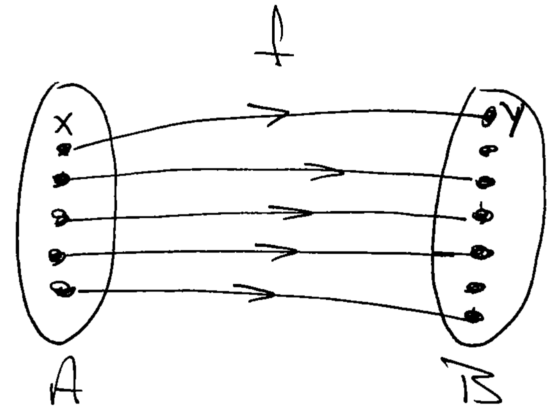
DEFN

A FUNCTION CONSISTS OF THREE THINGS:

(1) A SET A CALLED THE DOMAIN

(2) A SET B CALLED THE CODOMAIN

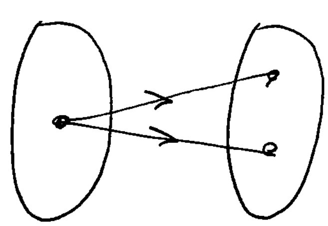
(3) A RULE f WHICH ASSIGNS TO EACH $x \in A$ A UNIQUE ELEMENT $y = f(x) \in B$.



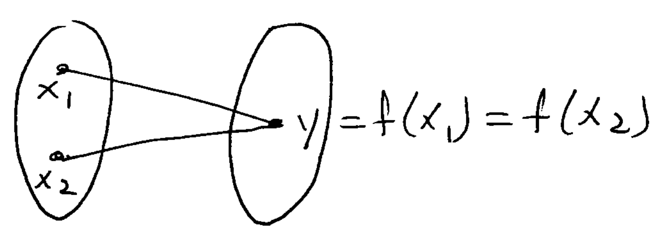
WE SAY $x \in A$ IS MAPPED TO $y = f(x) \in B$

y IS CALLED THE IMAGE OF x UNDER f

UNIQUENESS:



NOT A FUNCTION

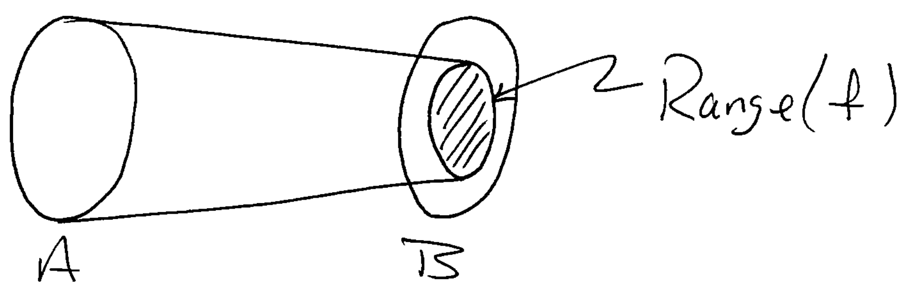


OK

NOTATION: WE WRITE $f: A \to B$ TO SAY f IS A FUNCTION WITH DOMAIN A AND CODOMAIN B.

THE RANGE OF f IS THE SET OF ALL IMAGES

$$\text{Range}(f) = \{ f(x) \in B \mid x \in A \} \subseteq B$$



NOTE $\text{Range}(f)$ NEED NOT BE ALL OF B. IF IT SO HAPPENS THAT

$$\text{Range}(f) = B$$

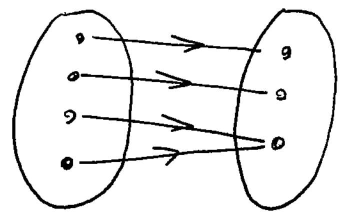
f IS SAID TO BE <u>ONTO</u> OR <u>SURJECTIVE</u>.

A FUNCTION f IS CALLED <u>ONE-TO-ONE</u> OR <u>INJECTIVE</u> IF NO TWO DISTINCT ELEMENTS IN A ARE MAPPED TO THE SAME ELEMENT IN B, i.e.

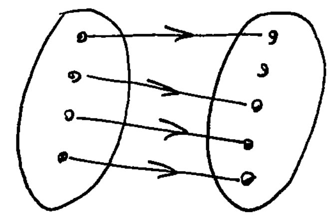
$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

EQUIVALENTLY

$$f(x_1) = f(x_2) \implies x_1 = x_2$$



NOT INJECTIVE



INJECTIVE

A FUNCTION WHICH IS BOTH INJECTIVE AND SURJECTIVE IS CALLED <u>BIJECTIVE</u> OR A <u>ONE-TO-ONE CORRESPONDANCE</u>

NOW SUPPOSE $f: A \to B$ IS BIJECTIVE. THEN WE CAN DEFINE A FUNCTION $g: B \to A$ AS FOLLOWS: ASSIGN TO EACH $y \in B$ THE UNIQUE $x \in A$ SUCH THAT $y = f(x)$, i.e. WE REQUIRE

$$x = g(y) \quad \text{IFF} \quad y = f(x).$$

NOTE SUCH AN $x \in A$ EXISTS SINCE f IS SURJECTIVE, AND IT IS UNIQUE SINCE f IS INJECTIVE.

THE FUNCTION $g: B \to A$ IS CALLED THE <u>INVERSE OF f</u>, AND IS DENOTED

$$g = f^{-1}$$

THUS WE SEE THAT A FUNCTION IS <u>INVERTIBLE</u> (i.e. HAS AN INVERSE) IFF IT IS BIJECTIVE.

<u>DEFN</u>

A <u>LINEAR TRANSFORMATION</u> $T: \mathbb{R}^m \to \mathbb{R}^n$ IS A FUNCTION OF THE FORM

$$T(\vec{x}) = A\vec{x}$$

WHERE A IS AN $n \times m$ MATRIX. WE CALL A THE MATRIX OF THE LINEAR TRANSFORMATION T.

<u>EX.</u> $$A = \begin{pmatrix} 3 & 1 & 2 & -1 \\ 0 & 1 & 7 & 0 \\ 4 & -5 & 0 & 2 \end{pmatrix} \qquad n=3,\ m=4.$$

THEN $T: \mathbb{R}^4 \to \mathbb{R}^3$ AND

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix},\ T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix},\ T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 0 \end{pmatrix}$$

AND $$T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

IN GENERAL THE A OF A LINEAR TRANSFORMATION T HAS COLUMNS

$$T(\vec{e}_1),\ T(\vec{e}_2),\ \ldots,\ T(\vec{e}_m)$$

WHERE $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$ ARE THE STANDARD BASIS VECTORS IN $\mathbb{R}^m$, i.e.

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{TH}} \text{ POSITION}$$

$m \times 1$

FOR $1 \le j \le m$. THUS A CAN ALWAYS BE RECOVERED FROM A KNOWLEDGE OF THE ACTION OF T ON $\vec{e}_1, \ldots, \vec{e}_m$!

$$A = \Big[\, T(\vec{e}_1)\ T(\vec{e}_2)\ \ldots\ T(\vec{e}_m) \,\Big]_{n \times m}$$

LINEAR TRANSFORMATIONS HAVE MANY SPECIAL PROPERTIES, SUCH AS

(1) $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$

(2) $T(k\vec{x}) = k\,T(\vec{x})$

FOR ANY $\vec{x}, \vec{y} \in \mathbb{R}^m$, $k \in \mathbb{R}$.

PROOF:

(1) $T(\vec{x}+\vec{y}) = A(\vec{x}+y) = A\vec{x} + A\vec{y}$
$= T(\vec{x}) + T(\vec{y})$.

(2) $T(k\vec{x}) = A(k\vec{x}) = k(A\vec{x})$
$= kT(\vec{x})$. ///

IN FACT LINEAR TRANSFORMATIONS CAN BE CHARACTERIZED BY PROPERTIES (1) AND (2).

THEOREM

A FUNCTION $T: \mathbb{R}^m \to \mathbb{R}^n$ IS A LINEAR TRANSFORMATION IFF (1) AND (2) HOLD FOR ALL $\vec{x}, \vec{y} \in \mathbb{R}^m$ AND $k \in \mathbb{R}$.

PROOF:

($\Longrightarrow$) DONE

($\Longleftarrow$) SUPPOSE $T: \mathbb{R}^m \to \mathbb{R}^n$ IS A FUNCTION WHICH SATISFIES (1) AND (2) FOR ALL $\vec{x}, \vec{y} \in \mathbb{R}^m$ AND $k \in \mathbb{R}$. WE MUST SHOW THAT THERE EXISTS A MATRIX A $(n \times m)$ SUCH THAT

$$T(\vec{x}) = A\vec{x} \quad \text{FOR ALL } \vec{x} \in \mathbb{R}^m .$$

LET $\vec{e}_1, \ldots, \vec{e}_m$ DENOTE THE STANDARD BASIS IN $\mathbb{R}^m$, AND DEFINE A AS

$$A = \left[T(\vec{e}_1)\, T(\vec{e}_2) \cdots T(\vec{e}_m) \right]$$

i.e. THE j^{th} COLUMN OF A IS THE VECTOR $T(\vec{e}_j) \in \mathbb{R}^n$ $(1 \le j \le m)$. THEN
$T(\vec{e}_1) = A\vec{e}_1$, $T(\vec{e}_2) = A\vec{e}_2$, $\ldots$, $T(\vec{e}_m) = A\vec{e}_m$.

NOW LET $\vec{x} \in \mathbb{R}^m$ BE CHOSEN ARBITRARILY, SAY

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_m \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$= x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_m \vec{e}_m .$$

THUS

By (1) $\quad T(\vec{x}) = T(x_1\vec{e}_1) + T(x_2\vec{e}_2) + \cdots + T(x_m\vec{e}_m)$

By (2) $\quad = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_m T(\vec{e}_m)$

$$= x_1 A\vec{e}_1 + x_2 A\vec{e}_2 + \cdots + x_m A\vec{e}_m$$

$$= A(x_1\vec{e}_1) + A(x_2\vec{e}_2) + \cdots + A(x_m\vec{e}_m)$$

$$= A(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_m\vec{e}_m)$$

$$= A\vec{x}$$

SINCE $\vec{x}$ WAS ARBITRARY, WE'VE SHOWN THAT $T(\vec{x}) = A\vec{x}$ FOR ANY $\vec{x} \in \mathbb{R}^m$. THUS T IS A LINEAR TRANSFORMATION WITH MATRIX A.

///

HW (2.1): 4, 6, 8, 12, 14, 16, 18, 20, 22, 26, 28, 34, 40, 42abc, 44, 46.