1.3 Counting Solutions

Consider a linear system of 3 equations in 3 unknowns x, y, z. By placing the augmented matrix (3×4) in RREF, we can see at a glance how many solutions the system has.

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ No solutions

$$\begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$$ Unique solution

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ Infinitely many solutions (line in $\mathbb{R}^3$)

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ Infinitely many solutions (Plane in $\mathbb{R}^3$)

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ Infinitely many solutions ($\mathbb{R}^3$)

DEFN

A LINEAR SYSTEM IS SAID TO BE CONSISTENT IF IT HAS AT LEAST ONE SOLUTION, OTHERWISE IT IS CALLED INCONSISTENT.

OBSERVE THAT A SYSTEM IS INCONSISTENT IFF THE RREF OF ITS AUGMENTED MATRIX CONTAINS THE ROW

$$\left(0 \ 0 \ \ldots \ 0 \,\middle|\, 1\right)$$

CORRESPONDING TO THE EQUATION $0=1$. IN PARTICULAR ONE CAN SEE THAT

$$\text{rank(AUG. MATRIX)} = \begin{cases} \text{rank(COEF. MATRIX)} & \text{IFF CONSISTENT} \\ \text{rank(COEF. MATRIX)} + 1 & \text{IFF INCONSISTENT} \end{cases}$$

ALSO NOTICE THAT A CONSISTENT SYSTEM MUST HAVE EITHER

- A UNIQUE SOLUTION, OR
- INFINITELY MANY SOLUTIONS.

AS WE SAW, THERE ARE MANY WAYS TO HAVE INFINITELY MANY SOLUTIONS.

MORE GENERALLY CONSIDER A LINEAR SYSTEM OF n EQUATIONS IN m UNKNOWNS.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n \end{cases}$$

THE COEFFICIENT MATRIX IS OF SIZE $n \times m$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

THE Right hand side is A $1 \times n$ Column VECTOR

$$\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

THE AUGMENTED MATRIX IS DENOTED $A|\vec{b}$ AND HAS SIZE $n \times (m+1)$

$$(A|b) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1m} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & b_n \end{array}\right)$$

- IF THE SYSTEM is CONSISTENT THEN

#cols OF (A|b) with leading 1 = rank(A|b) = rank(A)
cols of (A|b) without leading 1 = #free vars + 1
cols of (A|b) = #vars + 1

THUS #vars + 1 = (#free vars + 1) + rank(A)

$\therefore$ m = (#free variables) + rank(A)

- IF THE SYSTEM is INCONSISTENT THEN

#cols of (A|b) with leading 1 = rank(A|b) = rank(A) + 1
cols of (A|b) without leading 1 = #free vars
cols of (A|b) = #vars + 1

THUS #vars + 1 = #free vars + (rank(A) + 1)
AND AGAIN

$\therefore$ m = (#free variables) + rank(A)

WE SEE THAT IN EITHER CASE WE HAVE

$$\boxed{\#\text{free variables} = m - \text{rank}(A)}$$

THUS THE NUMBER OF SOLUTIONS IS MORE DIRECTLY RELATED TO THE RANK OF THE COEFFICIENT MATRIX THAN THAT OF THE AUGMENTED MATRIX.

SEVERAL FACTS ARE NOW APPARENT

(a) NECESSARILY $\text{rank}(A) \le n$ AND $\text{rank}(A) \le m$ SINCE THERE IS AT MOST ONE LEADING 1 IN EACH ROW AND COLUMN OF RREF(A).

(b) IF $\text{rank}(A) = n$, THEN THE SYSTEM IS CONSISTENT, SINCE IN THIS CASE THERE MUST BE A LEADING 1 IN EACH ROW OF RREF(A), WHENCE RREF(A|b) CANNOT CONTAIN THE ROW:

$$(0 \cdots\cdots 0 \mid 1)$$

(c) IF $\text{rank}(A) = m$, THEN THE SYSTEM HAS AT MOST ONE SOLUTION SINCE IN THIS CASE $\#$free variables $= m - \text{rank}(A) = 0$, HENCE THERE CANNOT BE INFINITELY MANY SOLUTIONS.

(d) IF $\text{rank}(A) < m$, THEN THE SYSTEM HAS EITHER NO SOLUTIONS OR INFINITELY MANY, SINCE $\#$free variables $= m - \text{rank}(A) > 0$, WHENCE IT CANNOT HAVE A UNIQUE SOLUTION

(e) IF $n < m$, THEN THE SYSTEM HAS EITHER NO SOLUTIONS OR INFINITELY MANY SOLUTIONS. THIS FOLLOWS FROM (a) AND (d) SINCE $\text{rank}(A) \leq n < m$.

(f) IF $n = m$, THEN THE SYSTEM HAS A UNIQUE SOLUTION IFF $\text{rank}(A) = n$

<u>PROOF</u>: IF $\text{rank}(A) < n = m$ THE SYSTEM CANNOT HAVE A UNIQUE SOLUTION BY (d). THUS UNIQUE SOLUTION $\Rightarrow \text{rank}(A) = n$. ON THE OTHER HAND IF $\text{rank}(A) = n = m$ THEN THE SYSTEM IS CONSISTENT BY (b), AND HAS AT MOST ONE SOLUTION BY (c), HENCE IT HAS A UNIQUE SOLUTION IN THIS CASE

$$\text{RREF}(A) = \begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdot & \cdot & 1 \end{pmatrix}$$

Summary

Let $r = \text{rank}(A)$. Then

case	# solns	examples	
1.) $r = m = n$	1 By (f)	$\left(\begin{array}{cc\|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right)$	$r = m = n = 2$
2.) $r = n < m$	∞ By (b), (d)	$\left(\begin{array}{ccc\|c} 1 & 0 & * & * \\ 0 & 1 & * & * \end{array}\right)$	$r = n = 2$, $m = 3$
3.) $r = m < n$	0, 1 By (e)	$\left(\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$ $\left(\begin{array}{cc\|c} 0 & 1 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{array}\right)$	$r = m = 2$, $n = 3$
4.) $r < m = n$	$0, \infty$ By (d)	$\left(\begin{array}{ccc\|c} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ $\left(\begin{array}{ccc\|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$	$r = 2$, $n = m = 3$
5.) $r < m < n$	$0, \infty$ By (d)	$\left(\begin{array}{ccc\|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ $\left(\begin{array}{ccc\|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)$	$r = 1$, $n = 2, m = 3$
6.) $r < n < m$	$0, \infty$ By (d)	$\left(\begin{array}{cc\|c} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$ $\left(\begin{array}{cc\|c} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$	$r = 1$, $m = 2, n = 3$

MATRIX OPERATIONS

LET A, B BE TWO $n \times m$ MATRICES.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{Am} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix}$$

WE DEFINE THEIR <u>SUM</u> TO BE

$$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & b_{nm}+b_{nm} \end{pmatrix}$$

LET $k \in \mathbb{R}$. WE DEFINE THE <u>SCALAR PRODUCT</u> kA TO BE THE $n \times m$ MATRIX

$$kA = Ak = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1m} \\ ka_{21} & ka_{22} & \cdots & ka_{2m} \\ \vdots & \vdots & & \vdots \\ ka_{n1} & ka_{n2} & \cdots & ka_{nm} \end{pmatrix}$$

AN EARLIER SOLUTION TO A LINEAR SYSTEM CAN NOW BE RE-WRITTEN:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7t - s + 3 \\ t \\ 2s + 2 \\ -s + 1 \\ s \end{pmatrix} = t\begin{pmatrix} 7 \\ 1 \\ 6 \\ 6 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

NOW LET A BE AN $n \times m$ MATRIX AND $\vec{x}$ BE AN $m \times 1$ MATRIX, i.e. A COLUMN VECTOR IN $\mathbb{R}^m$.

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

WE DEFINE THE PRODUCT $A\vec{x}$ TO BE THE $n \times 1$ MATRIX

$$A\vec{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{pmatrix}$$

OBSERVE THAT $A\vec{x}$ IS A COLUMN VECTOR IN $\mathbb{R}^n$. THUS A DEFINES A MAPPING FROM $\mathbb{R}^m$ TO $\mathbb{R}^n$

$$\vec{x} \in \mathbb{R}^m \longrightarrow A\vec{x} \in \mathbb{R}^n$$

LET $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ DENOTE THE COLUMNS OF A, i.e.

$$\vec{v}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix}, \quad \ldots, \vec{v}_m = \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

THEN THE PRODUCT $A\vec{x}$ CAN BE WRITTEN

$$A \cdot \vec{x} = \begin{pmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{n1}x_1 \end{pmatrix} + \begin{pmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{n2}x_2 \end{pmatrix} + \cdots + \begin{pmatrix} a_{1m}x_m \\ a_{2m}x_m \\ \vdots \\ a_{nm}x_m \end{pmatrix}$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + x_m \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix}$$

$$= x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_m \vec{v}_m .$$

WE SOMETIMES WRITE THIS AS

$$\left[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_m \right] \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

Ex
$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 3 & 4 & 7 \\ 1 & 9 & 0 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 14 \end{pmatrix}$$

$$(3 \times 4) \qquad (4 \times 1) \qquad (3 \times 1)$$

LIKEWISE

$$3\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 14 \end{pmatrix}$$

IN THE SPECIAL CASE WHEN $n=m$, A IS AN $n \times n$ (SQUARE) MATRIX AND REPRESENTS A TRANSFORMATION FROM $\mathbb{R}^n$ TO $\mathbb{R}^n$:

$$\vec{x} \in \mathbb{R}^n \longrightarrow A\vec{x} \in \mathbb{R}^n$$

WE DEFINE THE $n \times n$ <u>IDENTITY MATRIX</u> I TO HAVE 1'S ON ITS DIAGONAL AND 0'S ELSEWHERE.

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

I REPRESENTS THE IDENTITY MAPPING FROM $\mathbb{R}^n$ TO $\mathbb{R}^n$, i.e.

$$I\vec{x} = \vec{x} \quad \text{FOR ALL } \vec{x} \in \mathbb{R}^n$$

EX.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

DEFN

A LINEAR COMBINATION OF VECTORS $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$ IS AN EXPRESSION OF THE FORM

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m$$

WHERE $x_1, \dots, x_m \in \mathbb{R}$.

THUS THE PRODUCT $A\vec{x}$ IS A LINEAR COMBINATION OF THE COLUMNS OF A.

IN THE SPECIAL CASE $n=1$, THE MATRIX A IS A ROW VECTOR IN $\mathbb{R}^m$ AND THE PRODUCT $A\vec{x}$ IS A 1×1 MATRIX, i.e. A REAL NUMBER.

LET $\vec{a} = (a_1\ a_2 \cdots a_m) \in \mathbb{R}^m$ BE A ROW VECTOR AND LET

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$$

BE A COLUMN VECTOR. THE <u>DOT PRODUCT</u> OR <u>SCALAR PRODUCT</u> $\vec{a} \cdot \vec{x}$ IS DEFINED AS

$$\vec{a} \cdot \vec{x} = (a_1\ a_2 \cdots a_m) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = a_1 x_1 + a_2 x_2 + \cdots + a_m x_m .$$

RETURNING TO THE GENERAL CASE, LET $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n \in \mathbb{R}^m$ BE THE ROWS OF A, i.e.

$$\vec{w}_1 = (a_{11}\ a_{12} \cdots a_{1m})$$
$$\vec{w}_2 = (a_{21}\ a_{22} \cdots a_{2m})$$
$$\vdots$$
$$\vec{w}_n = (a_{n1}\ a_{n2} \cdots a_{nm})$$

THEN

$$A\vec{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{pmatrix} = \begin{pmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{pmatrix}$$

i.e. THE ENTRIES OF THE COLUMN VECTOR $A\vec{x} \in \mathbb{R}^n$ ARE THE DOT PRODUCTS OF EACH OF THE n ROWS OF A WITH $\vec{x}$. SOMETIMES WE WRITE

$$A\vec{x} = \begin{bmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vdots \\ \vec{w}_n \end{bmatrix} \vec{x} = \begin{pmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{pmatrix}$$

<u>THEOREM</u>

LET A, B BE $n \times m$ MATRICES, $\vec{x}, \vec{y} \in \mathbb{R}^m$ BE COLUMN VECTORS, AND $k \in \mathbb{R}$. THEN

a.) $A(\vec{x} + \vec{y}) = A\vec{x} + B\vec{y}$

b.) $A(k\vec{x}) = \vec{k}(A\vec{x})$

c.) $(A+B)\vec{x} = A\vec{x} + B\vec{x}$

WE DEFER THE PROOF UNTIL LATER.

WITH OUR MATRIX OPERATIONS WE CAN WRITE A LINEAR SYSTEM

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1 \\ a_{21}x_2 + a_{22}x_2 + \cdots + a_{2m}x_m = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n \end{cases}$$

AS

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

i.e.

$$A\vec{x} = \vec{b}$$

SOLVING THIS SYSTEM THEN MEANS FINDING ALL VECTORS $\vec{x} \in \mathbb{R}^m$ SUCH THAT $A\vec{x} = \vec{b}$ WHERE $\vec{b} \in \mathbb{R}^n$.

NOTICE THIS IS ANALOGOUS TO SOLVING THE SCALAR EQUATION $ax = b$. WE KNOW THAT IF $a \neq 0$, THE SOLUTION IS $x = \frac{b}{a}$, AND IF $a = 0$, THE EQUATION HAS A SOLUTION ONLY IF $b = 0$, IN WHICH CASE ANY $x \in \mathbb{R}$ IS A SOLUTION.

WE WISH TO FIND A WAY TO "DIVIDE" THE EQUATION $A\vec{x} = \vec{b}$ BY A, WHATEVER THAT MAY MEAN.

HW (1.3) 2, 4, 6, 10, 12, 14, 16, 18, 20ab, 22, 24, 26, 28, 46, 50.