CHAPTER 8 DIFFERENTIAL EQUATIONS

THE GENERAL FIRST ORDER ORDINARY DIFFERENTIAL EQUATION (ODE) IS

$$\frac{dy}{dx} = F(x,y)$$

A SOLUTION TO THIS EQUATION ON AN INTERVAL $I \subseteq \mathbb{R}$ IS A FUNCTION $y = y(x)$ SATISFYING

$$y'(x) = F(x, y(x)) \quad \text{FOR ALL } x \in I$$

THERE IS A LARGE AND WELL DEVELOPED THEORY CONCERNING SUCH PROBLEMS. WE RESTRICT OUR ATTENTION TO SO CALLED SEPARABLE EQUATIONS.

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

AND SPECIAL CASES

$$\frac{dy}{dx} = f(x) \qquad (\text{i.e. } g(y) = 1)$$

$$\frac{dy}{dx} = g(y) \qquad (\text{i.e. } f(x) = 1)$$

OFTEN WE WILL TAKE THE INDEPENDENT VARIABLE x TO BE TIME, SO WRITE t INSTEAD OF x. ALSO, OFTEN THE DEPENDENT VARIABLE WILL BE SOMETHING OTHER THAN y, AND HAVE SOMETHING LIKE $N = N(t)$, i.e. PROBLEMS LIKE

$$\frac{dN}{dt} = f(t) \quad \text{or} \quad \frac{dN}{dt} = g(N)$$

(8.1.1) PURE TIME ODEs

$$\frac{dy}{dx} = f(x) \qquad \text{FOR } x \in I$$

NOTICE THIS IS JUST AN INTEGRATION PROBLEM, i.e.

$$y(x) = \int_{x_0}^{x} f(u)\,du + C$$

WHERE $x_0 \in I$. THIS FOLLOWS FROM THE FUNDAMENTAL THEOREM OF CALCULUS (FIRST VERSION.)

OBSERVE THAT

$$Y_0 \overset{\text{def}}{=} y(x_0) = 0 + C = C .$$

THUS

$$y(x) = Y_0 + \int_{x_0}^{x} f(u)du$$

IS THE UNIQUE SOLUTION TO THE INITIAL VALUE PROBLEM (IVP)

$$\begin{cases} \dfrac{dy}{dx} = f(x) & x \in I \\ y(x_0) = Y_0 \end{cases}$$

NOTE THAT AN ODE LIKE $\frac{dy}{dx} = f(x)$ IN GENERAL HAS INFINITELY MANY SOLUTIONS. A SINGLE SOLUTION IS SPECIFIED ONLY WHEN THE INITIAL VALUE $y(x_0)$ IS GIVEN.

Ex. Solve the IVP.

$$\begin{cases} \dfrac{dV}{dt} = \sin t \\ V(0) = 3 \end{cases}$$

Soln: $V(t) = 4 - \cos t$

Ex. Solve the IVP.

$$\begin{cases} \dfrac{dw}{dr} = \ln r \\ w(1) = 4 \end{cases}$$

Soln: $w(r) = r\ln r - r + 5$