7.7.3 THE ERROR TERM

LET $f(x), f'(x), \ldots, f^{(n)}(x), f^{(n+1)}(x)$ BE DEFINED AND CONTINUOUS AT $a \in I$, WHERE I IS AN INTERVAL.

LET $P_n(x)$ BE THE n^{TH} DEGREE TAYLOR POLYNOMIAL FOR $f(x)$ AT $x=a$, i.e.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

DEFINE FOR $x \in I$:

$$R_{n+1}(x) = f(x) - P_n(x)$$

i.e. $R_{n+1}(x)$ IS THE ERROR INCURRED BY REPLACING $f(x)$ BY $P_n(x)$

$R_{n+1}(x)$ IS CALLED THE $(n+1)^{ST}$ <u>ERROR TERM</u> OR <u>REMAINDER TERM</u>.

<u>THEOREM</u>

THERE EXISTS c BETWEEN x AND a SUCH THAT

$$\boxed{R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$$

THIS EXPRESSION CAN BE USED TO PLACE BOUNDS ON THE ERROR IN THE APPROXIMATION

$$f(x) \approx P_n(x)$$

FOR $x \in I$.

EX. $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$\therefore$ $R_{n+1}(x) = \frac{e^c}{(n+1)!} x^{n+1}$ FOR SOME

c BETWEEN x AND 0, SINCE

$$\left.\frac{d^{n+1}}{dx^{n+1}}\left[e^x\right]\right|_{x=c} = e^c.$$

SUPPOSE WE WISH TO APPROXIMATE e^x FOR $x \in (-1,1)$ TO 6 DECIMAL PLACES BY THE TAYLOR POLYNOMIAL

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

HOW BIG SHOULD n BE ?

WE HAVE

$$|R_{n+1}(x)| = \frac{e^c}{(n+1)!}|x|^{n+1} \le \frac{e}{(n+1)!}$$

SINCE $x \in (-1,1) \Rightarrow |x| \le 1$, AND c BETWEEN x AND 0 IMPLIES

$$-1 < c < 1 \Rightarrow e^{-1} < e^c < e^1 = e .$$

THUS BY CHOOSING n TO SATISFY

$$\frac{e}{(n+1)!} \le 10^{-6}$$

WE WILL GUARANTEE $|R_{n+1}(x)| \le 10^{-6}$ (FURTHERMORE WE SEEK THE SMALLEST SUCH n.) ONE CHECKS THAT

$$9! < 10^6 \cdot e < 10!$$

$$\therefore \frac{e}{10!} < 10^{-6} < \frac{e}{9!}$$

THUS WE CHOOSE $n+1 = 10$, i.e. $\boxed{n = 9}$.

THEREFORE THE APPROXIMATION

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^9}{9!}$$

WILL NEVER HAVE ERROR $> 10^{-6}$ AS LONG AS $x \in (-1,1)$.

EXERCISE

APPROXIMATE e^x FOR $x \in (-2,2)$ WITH ERROR $\leq 10^{-4}$.

HW 9: 7.7.4 (P. 469)

2-16 even

20, 22, 26ab, 28, 30