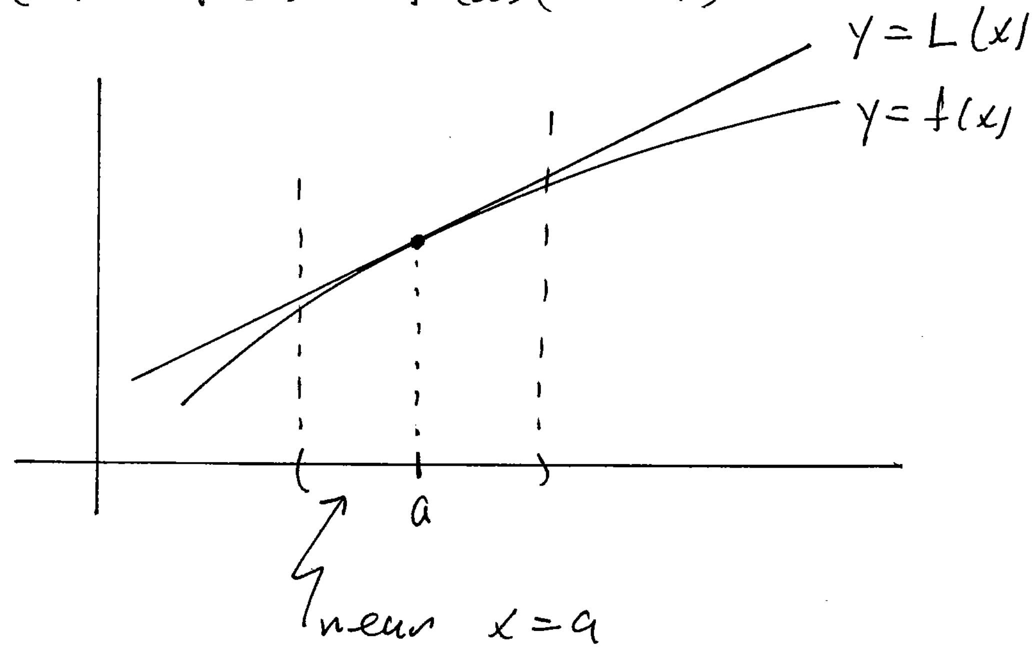
7.7.2 Taylor Polynomial at $x=a$

Recall the linear approximation to $f(x)$ at $x=a$ is given by

$$L(x) = f(a) + f'(a)(x-a)$$



To get the n^{th} degree Taylor Polynomial at $x=a$ we likewise evaluate f and its derivatives at $x=a$, and shift x to $(x-a)$.

i.e. $f(x) \approx \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

THUS

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^n$$

IS THE UNIQUE POLYNOMIAL OF DEGREE n SATISFYING THE $n+1$ CONDITIONS

$$P_n(a) = f(a)$$
$$P_n'(a) = f'(a)$$
$$\vdots$$
$$P_n^{(k)}(a) = f^{(k)}(a)$$
$$\vdots$$
$$P_n^{(n)}(a) = f^{(n)}(a)$$

Ex. $n=3$, $a=2$, $f(x)=e^x$.

$$P_3(x) = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3$$

OBSERVE THAT THIS APPROXIMATION CAN ALSO BE DERIVED FROM THE TAYLOR POLYNOMIAL OF e^x ABOUT $x=0$ (INSTEAD OF $x=2$) AS FOLLOWS:

$$e^x = e^{2+(x-2)} = e^2 \cdot e^{x-2}$$

$$\approx e^2\left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!}\right)$$

$$= e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 .$$

NOTE: THE SYMBOL '$\approx$' HERE STANDS FOR 'APPROXIMATELY EQUAL TO'.

EX. RECALL THE TAYLOR POLYNOMIAL FOR $\frac{1}{1-x}$ AT $x=0$ IS (EXERCISE)

$$\frac{1}{1-x} \approx 1 + x + x^2 + \cdots + x^n$$

WE CAN USE THIS TO FIND THE TAYLOR POLYNOMIAL FOR $\frac{1}{x}$ AT $x=1$ AS FOLLOWS

$$\frac{1}{x} = \frac{1}{1-(1-x)} \approx 1 + (1-x) + (1-x)^2 + \cdots + (1-x)^n$$

$$= 1 - (x-1) + (x-1)^2 - \cdots + (-1)^n (x-1)^n$$