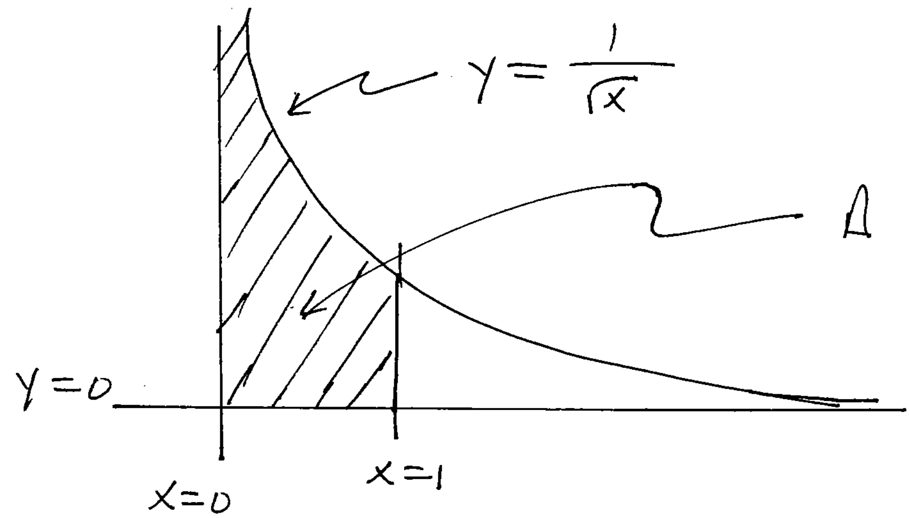
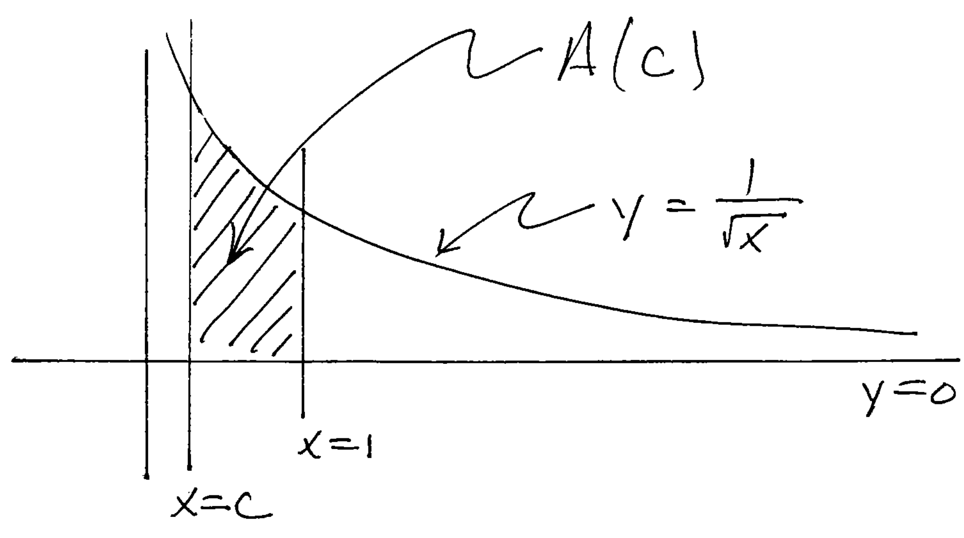
(7.4.2) UNBOUNDED INTEGRANDS

EX. FIND THE AREA BOUNDED BY $y = 1/\sqrt{x}$, $y=0$, $x=0$, $x=1$



AS WE CAN SEE, THE REGION IS UNBOUNDED.

RELATED PROBLEM: FIND THE AREA BOUNDED BY $y = 1/\sqrt{x}$, $y=0$, $x=1$, $x=c$ WHERE $0<c<1$.



$$A(c) = \int_c^1 x^{-1/2}\,dx = 2x^{1/2}\Big|_c^1 = 2(1-\sqrt{c})$$

THUS WE DEFINE

$$A = \lim_{c \to 0^+} A(c) = \lim_{c \to 0^+} 2(1-\sqrt{c})$$

$$= 2.$$

MORE GENERALLY IF $\lim\limits_{x \to a^+} f(x) = \pm\infty$,

WE DEFINE

$$\boxed{\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx}$$

AND IF $\lim\limits_{x \to b^-} f(x) = \pm\infty$, WE DEFINE

$$\boxed{\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx}$$

IF THESE LIMITS EXIST WE SAY THE INTEGRAL <u>CONVERGES</u>, OTHERWISE IT <u>DIVERGES</u>

IF f IS UNBOUNDED AT BOTH ENDPOINTS OF $[a,b]$ WE PICK ANY POINT

$$p \in (a,b)$$

AND DEFINE

$$\boxed{\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx}$$

THIS INTEGRAL IS SAID TO CONVERGE, PROVIDED BOTH IMPROPER INTEGRALS ON THE RIGHT CONVERGE (INDEPENDENTLY).

IF $f(x)$ IS UNBOUNDED AT SOME INTERIOR POINT $d \in (a,b)$ THEN WE DEFINE AGAIN

$$\boxed{\int_a^b f(x)\,dx = \int_a^d f(x)\,dx + \int_d^b f(x)\,dx}$$

WHERE IT IS AGAIN REQUIRED THAT THE TWO INTEGRALS ON THE RIGHT CONVERGE IN ORDER FOR THE ONE ON THE LEFT TO CONVERGE.

Examples

$$\int_{-1}^{1} \frac{1}{x^2}\,dx \quad \text{(DIVERGES)}$$

$$\int_{0}^{1} \ln x\,dx \quad \text{(CONVERGES)}$$

$$\int_{0}^{2} (x-1)^{-1/3}\,dx \quad \text{(CONVERGES)}$$

EX. FOR WHICH p DOES $\int_0^1 \frac{1}{x^p}\,dx$ CONVERGE?

$$\boxed{\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p} & \text{IF } p<1 \\ \infty & \text{IF } p \geq 1 \end{cases}}$$

TO PROVE THIS, BREAK INTO FOUR CASES:

1. $p \leq 0$: NOT IMPROPER
2. $0 < p < 1$: CONVERGES
3. $p = 1$: DIVERGES
4. $p > 1$: DIVERGES