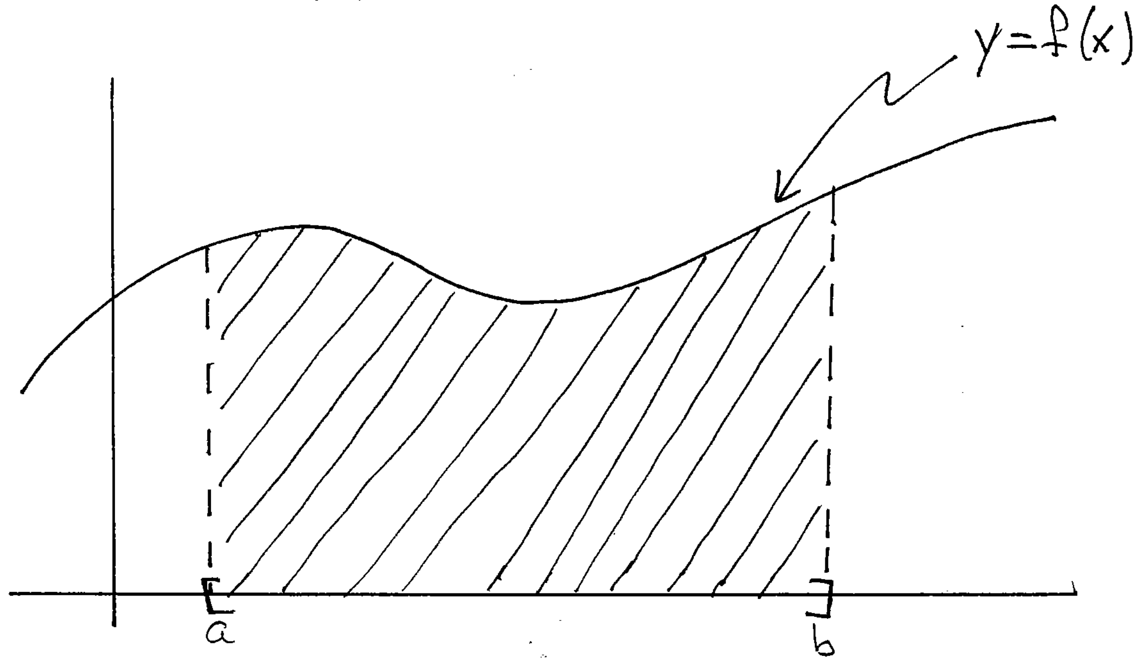
6.1 THE DEFINITE INTEGRAL

6.1.1 AREA

WE ARE CONCERNED WITH THE FOLLOWING PROBLEM FROM GEOMETRY. GIVEN A CONTINUOUS FUNCTION

$$f : [a, b] \to \mathbb{R}$$

DETERMINE THE AREA UNDER THE GRAPH OF f FROM a TO b.

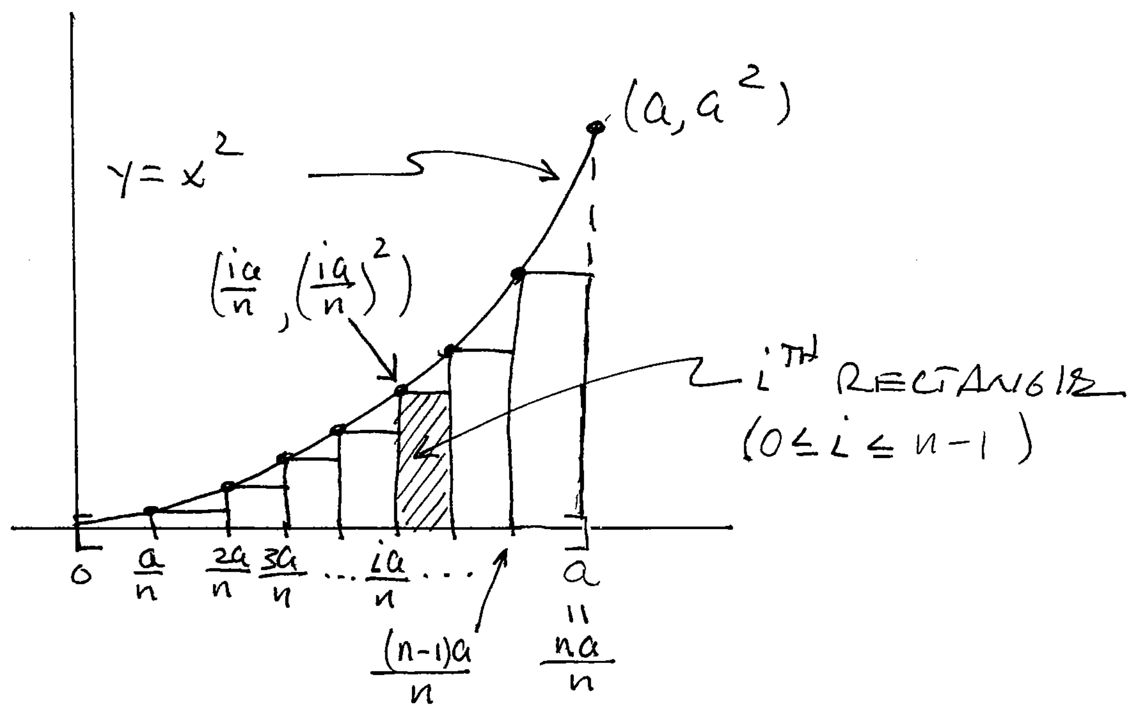


TWO FACTS WHICH MAY NOT BE IMMEDIATELY OBVIOUS.

- THIS PROBLEM IS CLOSELY RELATED TO ANTIDIFFERENTIATION

- THIS PROBLEM IS INTIMATELY CONNECTED TO MANY PROBLEMS IN THE PHYSICAL AND BIOLOGICAL SCIENCES.

EX FIND THE AREA UNDER $y = x^2$ FROM $x = 0$ TO $x = a$



WE FIRST CALCULATE AN APPROXIMATION AS FOLLOWS. PARTITION $[0, a]$ INTO n SUB-INTERVALS OF EQUAL WIDTH:

$$\left[0, \frac{a}{n}\right], \left[\frac{a}{n}, \frac{2a}{n}\right], \left[\frac{2a}{n}, \frac{3a}{n}\right], \ldots\ldots, \left[\frac{(n-1)a}{n}, a\right]$$

THEN ERECT A RECTANGLE OVER EACH SUBINTERVAL WHICH IS INSCRIBED IN THE CURVE $y = x^2$.

WE THEN SUM THE AREAS OF THESE n RECTANGLES AND LET THAT SUM, DENOTED S_n, SERVE AS OUR APPROXIMATION.

THE i^{TH} RECTANGLE HAS

$$\text{height} = \left(\frac{ia}{n}\right)^2$$
$$\text{width} = \frac{a}{n}$$
$$\therefore \text{ area} = \frac{i^2 a^3}{n^3}$$

FOR $i = 0, 1, \ldots, n-1$. THUS

$$S_n = \sum_{i=0}^{n-1} \frac{i^2 \cdot a^3}{n^3} = \left(\frac{a^3}{n^3}\right) \cdot \sum_{i=1}^{n-1} i^2$$

A USEFUL FACT:

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

WHENCE

$$\sum_{i=1}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6}$$

SO THAT

$$S_n = \left(\frac{a^3}{n^3}\right)\left(\frac{n(n-1)(2n-1)}{6}\right)$$

$$= \frac{a^3}{6} \cdot \frac{n}{n} \cdot \left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right)$$

$$= \frac{a^3}{6} \cdot \left(1 - \frac{1}{n}\right)\left(2 - \frac{2}{n}\right)$$

NOW OBSERVE THAT OUR APPROXIMATION S_n BECOMES BETTER AS n IS INCREASED. THE AREA S UNDER $y = x^2$ FROM $x = 0$ TO $x = a$ SHOULD BE THE LIMIT OF THESE APPROXIMATIONS AS $n \to \infty$. THUS

$$S = \lim_{n \to \infty} S_n$$

$$= \lim_{n \to \infty} \frac{a^3}{6} \cdot \left(1 - \frac{1}{n}\right)\left(2 - \frac{2}{n}\right)$$

$$= \frac{a^3}{6} \cdot (1 - 0)(2 - 0) = \boxed{\frac{a^3}{3}}$$

FOR THE MOMENT, DEFINE

$$S(x) = \frac{1}{3} x^3$$

AND OBSERVE THAT $S = S(a)$. BUT ALSO

$$S'(x) = x^2$$

SO THAT $S(x)$ IS AN ANTIDERIVATIVE OF THE FUNCTION $f(x) = x^2$, WHICH BOUNDS THE AREA S. THIS IS NO ACCIDENT.

WE WISH TO GENERALIZE THIS TECHNIQUE AS FAR AS POSSIBLE. BEFORE DOING SO WE REVIEW A FEW FACTS ABOUT FINITE SUMS.

NOTATION

LET $a_1, a_2, \ldots, a_n$ BE A FINITE SEQUENCE OF REAL NUMBERS. WE WRITE

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

NOTE ALSO

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_k$$

i.e. THE INDEX VARIABLE i (OR k) IS A DUMMY VARIABLE, i.e. A BOOKKEEPING DEVICE WHICH DOES NOT APPEAR IN THE 'ANSWER'.

THM

(1) $\sum_{i=1}^{n} c \cdot a_i = c \cdot \sum_{i=1}^{n} a_i$

(2) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

(3) $\sum_{i=1}^{n} 1 = n$

(4) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(5) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(6) $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

(7) $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$

THERE IS A GENERAL FORMULA FOR THE SUM $\sum_{i=1}^{n} i^k$ WHICH IS OUTSIDE THE SCOPE OF THIS COURSE.

PROOF OF (4):

LET
$$S = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

$$\therefore \quad S = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

$$\therefore \quad S + S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)$$

$$\therefore \quad 2S = n(n+1).$$

HENCE $S = \frac{n(n+1)}{2}$, AS CLAIMED.

EXERCISE

USE THE SAME TECHNIQUE TO SHOW THAT

$$\sum_{i=1}^{n} (2i-1) = n^2$$

i.e. $1 + 3 + 5 + \cdots + (2n-1) = n^2$, i.e. THE SUM OF THE FIRST n ODD POSITIVE INTEGERS IS n^2.

WHAT ABOUT THE SUM OF THE FIRST n EVEN POSITIVE INTEGERS.