PRACTICE MIDTERM I
SOLUTIONS

1 a. EXPRESS RIEMANN SUM AS A DEFINITE INTEGRAL FROM $[-\pi, 3\pi]$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin(x_k)}{1+x_k} \Delta x_k$$

$\Rightarrow$ USE DEFINITION:

$$\int_a^b f(x)\,dx \equiv \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

SO WE HAVE

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin(x_k)}{1+x_k} \Delta x_k = \boxed{\int_{-\pi}^{3\pi} \frac{\sin(x)}{1+x}\,dx}$$

b. EXPRESS RIEMANN SUM AS AN INTEGRAL

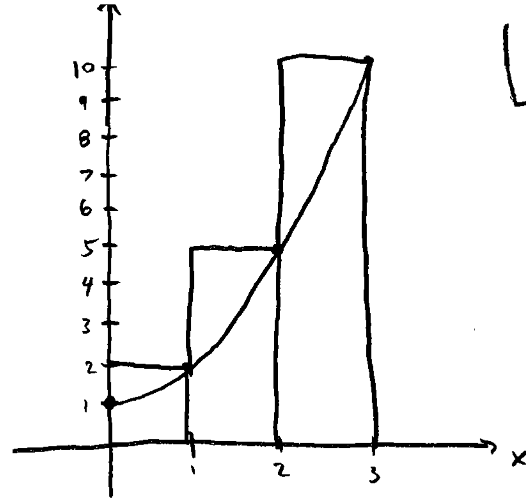
$$\int_{-3}^{5} \frac{x^2}{1+x^3}\,dx$$

$\Rightarrow$ USE SAME DEFINITION FROM PART (a)

AND WE GET

$$\int_{-3}^{5} \frac{x^2}{1+x^3}\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} \frac{(c_i)^2}{1+(c_i)^3} \Delta x_i$$

ON THE INTERVAL $[-3, 5]$



(2) a) RIEMANN SUM:

- $n = 3$
- $y = x^2 + 1$
- $0 \leq x \leq 3$
- RIGHT ENDPOINTS

$$\Delta x_i = \frac{b-a}{n} = \frac{3-0}{3} = 1$$

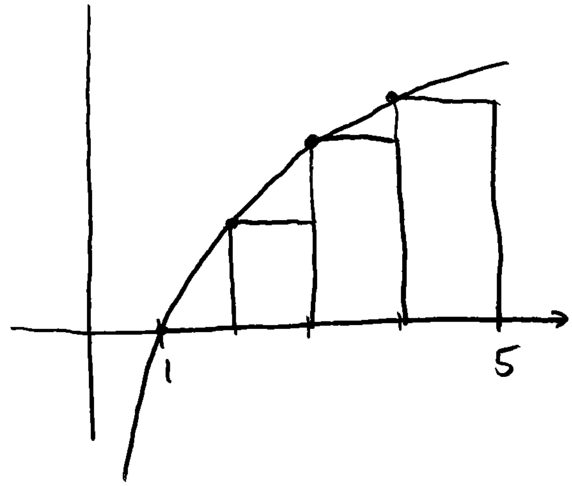
i	c_i	$f(c_i)$	Δx_i	A_i
1	1	2	1	2
2	2	5	1	5
3	3	10	1	10

$$\int_0^3 (x^2+1)\,dx \approx \sum_{i=1}^{3} f(c_i)\,\Delta x_i \quad = \text{AREA (APPROX.)} = A_1 + A_2 + A_3$$

$$\Rightarrow \int_0^3 (x^2+1)\,dx \approx 17$$

2b) RIEMANN SUM

- $n = 4$
- $y = \ln x$
- $1 \leq x \leq 5$
- LEFT ENDPOINTS



$$\int_1^5 \ln(x)\,dx \approx \sum_{i=1}^{5} \ln(c_i)\,\Delta x_i$$

$$\Delta x_i = \frac{b-a}{n} = \frac{4}{4} = 1$$

i	c_i	$f(c_i)$	Δx_i	A_i
1	1	$\ln(1) = 0$	1	0
2	2	$\ln(2)$	1	$\ln 2$
3	3	$\ln(3)$	1	$\ln 3$
4	4	$\ln(4)$	1	$\ln 4$

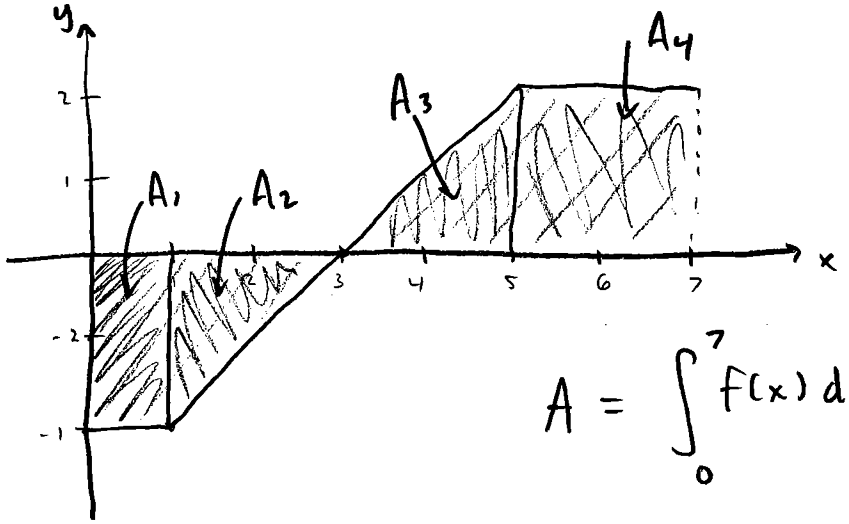
APPROXIMATE AREA $= 0 + \ln 2 + \ln 3 + \ln 4$

$= \ln(2 \cdot 3 \cdot 4) = \ln(24)$

* RECALL:

$$\ln(a) + \ln(b) = \ln(ab)$$

3) a)

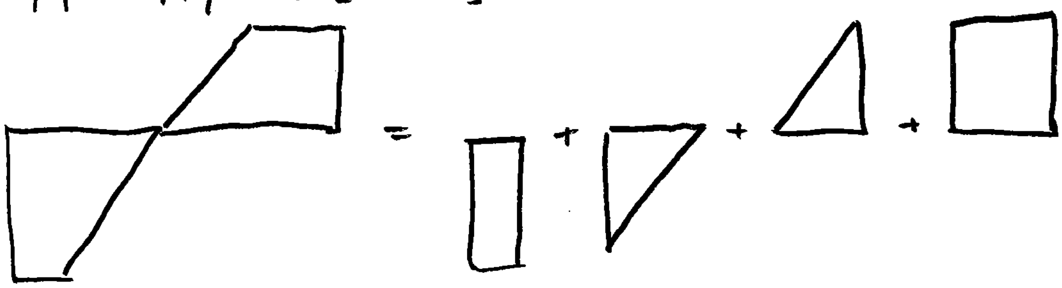


$$A = \int_0^7 f(x)\,dx = ?$$

WHAT IS THIS AREA?

$\Rightarrow$ USE SIMPLE GEOMETRIC SHAPES...

$$A = A_1 + A_2 + A_3 + A_4$$



$$= (b\cdot h) + \left(\tfrac{1}{2}b\cdot h\right) + \tfrac{1}{2}(b\cdot h) + (b\cdot h)$$

$$= 1\cdot(-2) + \tfrac{1}{2}\cdot 2(-2) + \tfrac{1}{2}2\cdot 2 + 2\cdot 2$$

$$-2 - 2 + 2 + 4 = \boxed{2 = A}$$

b) AVERAGE VALUE OF $f(x)$ ON $[0,7]$

$\Rightarrow$ RECALL: $\text{AVERAGE}(f(x)) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

SO THE AVERAGE VALUE OF "OUR" $f(x)$ IS

$$\frac{1}{7-0}\int_0^7 f(x)\,dx = \frac{1}{7}(2) = \boxed{\frac{2}{7}}$$

* WE GOT $\int_0^7 f(x)\,dx$ FROM PART a...

(4) $f(x) = \ln(x)$ REPRESENTS THE PATH OF A PARTICLE

a) **SETUP** INTEGRAL DESCRIBING LENGTH OF PATH: FROM $[0, t]$:

RECALL:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx \quad \text{(ARCLENGTH FORMULA)}$$

SO WE JUST "BUILD" THE DEFINITION OF ARCLENGTH:

$$f(x) = \ln x$$

$$f'(x) = 1/x$$

$$\Rightarrow L = \int_0^t \sqrt{1 + \left(\tfrac{1}{x}\right)^2}\, dx$$

b) FIND RATE OF CHANGE OF L:

$$R(t) = \frac{d}{dt} L(t) = \frac{d}{dt} \int_0^t \sqrt{1 + \left(\tfrac{1}{x}\right)^2}\, dx$$

RECALL: FTC II: $\frac{d}{dx} \int_{g(x)}^{h(x)} f(u)\, du = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x$

SO: $R(t) = \sqrt{1 + \left(\tfrac{1}{t}\right)^2}$ ✓

5 FIND ANTIDERIVATIVES:

a) $f(x) = \dfrac{x^2+1+x}{(x^2+1)x} = \dfrac{\cancel{(x^2+1)}}{\cancel{(x^2+1)}x} + \dfrac{\cancel{x}}{(x^2+1)\cancel{x}}$

$$f(x) = \frac{1}{x} + \frac{1}{x^2+1}$$

WE WANT $\int f(x)\,dx$

RECALL: $\int \frac{1}{x}\,dx = \ln|x| + c$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

SO WE HAVE

$$\int f(x)\,dx = \int\left(\frac{1}{x} + \frac{1}{x^2+1}\right)dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$\boxed{\int f(x)\,dx = \ln|x| + \tan^{-1}(x) + C}$$

b) $g(x) = \dfrac{1}{2+32x^2}$

$$\int g(x)\,dx = \int \frac{1}{2+32x^2}\,dx = \int \frac{1}{2(1+16x^2)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{1+16x^2}\,dx = \frac{1}{2}\int \frac{1}{1+(4x)^2}\,dx$$

let $u = 4x$
$du = 4\,dx$
$dx = \dfrac{du}{4}$

PLUG THESE IN

$$\Rightarrow \int g(x)\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,\frac{du}{4} = \frac{1}{8}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{8}\tan^{-1}(u) + C$$

$$\Rightarrow \boxed{\int g(x)\,dx = \frac{1}{8}\tan^{-1}(4x) + C}$$

c) $h(x) = \dfrac{1-x^2}{\sqrt{(1-x^2)^3}} + \csc x \cot x$

$$h(x) = \frac{(1-x^2)}{((1-x^2)^3)^{1/2}} + \frac{1}{\sin x}\,\frac{\cos x}{\sin x}$$

$$h(x) = \frac{(1-x^2)}{(1-x^2)^{3/2}} + \frac{\cos x}{\sin^2 x}$$

RECALL $\frac{x^a}{x^b} = x^{a-b}$ OR $x^a \cdot x^b = x^{a+b}$

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$$h(x) = (1-x^2)^{(1-3/2)} + \frac{\cos x}{\sin^2 x}$$

$$h(x) = \frac{1}{(1-x^2)^{1/2}} + \frac{\cos x}{\sin^2 x}$$

SO $$\int h(x)\,dx = \int \frac{1}{(1-x^2)^{1/2}}\,dx + \int \frac{\cos x}{\sin^2 x}\,dx$$

RECALL $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

SO $$\int h(x)\,dx = \sin^{-1} x + \int \frac{\cos x}{(\sin x)^2}\,dx$$

let $u = \sin x$ PLUG THESE IN

$du = \cos x\,dx$

$dx = \frac{du}{\cos x}$

$$\Rightarrow \int h(x)\,dx = \sin^{-1} x + \int \frac{\cancel{\cos x}}{u^2}\,\frac{du}{\cancel{\cos x}} = \sin^{-1} x + \int \frac{1}{u^2}\,du$$

$$\int h(x)\,dx = \sin^{-1} x - u^{-1} + C$$

$$\Rightarrow \boxed{\int h(x)\,dx = \sin^{-1} x - \frac{1}{\sin x} + C}$$

d) $p(x) = (3e^{x})e^{2x} + e^{x}e^{-x} + \frac{5}{2}(e^{-x})^{5}$

$$P(x) = 3e^{x+2x} + e^{x-x} + \frac{5}{2}e^{(-x)\cdot 5}$$

$$P(x) = 3e^{3x} + 1 + \frac{5}{2}e^{-5x}$$

So

$$\int P(x)\,dx = \int \left(3e^{3x} + 1 + \frac{5}{2}e^{-5x}\right)dx$$

$$= 3\int e^{3x}\,dx + \int 1\,dx + \frac{5}{2}\int e^{-5x}\,dx$$

$$= 3\left[\frac{1}{3}e^{3x}\right] + x + \frac{5}{2}\left[-\frac{1}{5}e^{-5x}\right] + c$$

$$\Rightarrow \boxed{\int P(x)\,dx = e^{3x} + x - \frac{1}{2}e^{-5x} + c}$$

(6) FIND THE LENGTH OF

$$y^{1/3} = \sqrt{x^2 + \frac{6}{9}}$$

FROM 0 TO 1.

$\Rightarrow$ USE ARCLENGTH FORMULA:

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

i) NEED TO FIND $f(x) = y$

$$y^{1/3} = \left(x^2 + \frac{6}{9}\right)^{1/2}$$

$$\left(y^{1/3}\right)^3 = \left[\left(x^2 + \frac{6}{9}\right)^{1/2}\right]^3$$

$$\rightarrow y = \left(x^2 + \frac{6}{9}\right)^{3/2} \Rightarrow y' = f'(x) = \frac{3}{2}\left(x^2 + \frac{6}{9}\right)^{1/2} \cdot 2x$$

$$f'(x) = 3x\left(x^2 + \frac{6}{9}\right)^{1/2}$$

ii) PLUG INTO L

$$L = \int_0^1 \sqrt{1 + \left(3x\left(x^2 + \frac{6}{9}\right)^{1/2}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 9x^2\left(x^2 + \frac{6}{9}\right)}\, dx = \int_0^1 \sqrt{9x^4 + 6x^2 + 1}\, dx$$

NOW: $9x^4 + 6x^2 + 1 = (3x^2 + 1)(3x^2 + 1) = (3x^2 + 1)^2$

SO: $L = \int_0^1 \sqrt{(3x^2+1)^2}\, dx = \int_0^1 (3x^2 + 1)\, dx$

$$\Rightarrow L = \int_0^1 3x^2\,dx + \int_0^1 1\,dx$$

$$= x^3\Big|_0^1 + x\Big|_0^1$$

$$= \left((1)^3 - (0)^3\right) + (1 - 0)$$

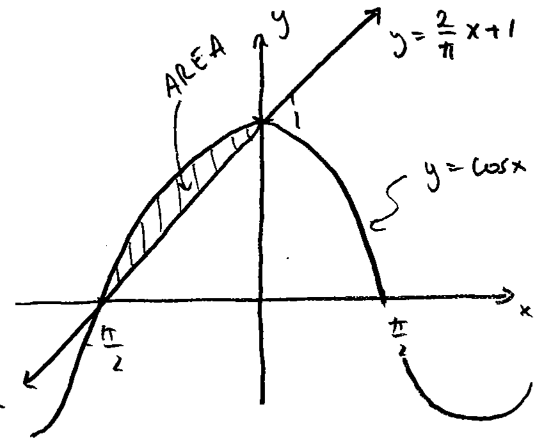
$$= 1 + 1 = \boxed{2 = L}$$

(7) FIND AREA BOUNDED BY

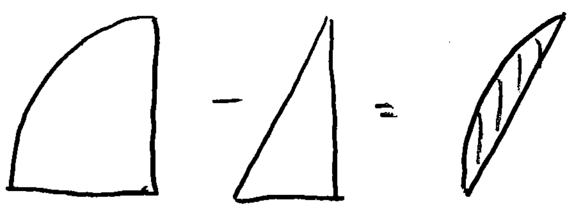
$y = \cos x$

$y = \frac{2}{\pi}x + 1$

IN THE 2^{ND} QUADRANT



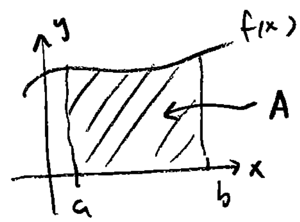
$\Rightarrow$ GENERAL IDEA

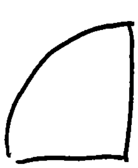
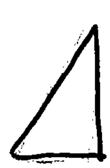


= AREA WE WANT

RECALL WHAT AN INTEGRAL (DEFINITE) IS

$$\int_a^b f(x)\,dx = A$$



SO  $= \int_{-\frac{\pi}{2}}^{0} \cos x\,dx$,  $= \int_{-\frac{\pi}{2}}^{0} \left(\frac{2}{\pi}x + 1\right) dx$

SO  = A

IS $A = \int_{-\frac{\pi}{2}}^{0} \cos x \, dx - \int_{-\frac{\pi}{2}}^{0} \left(\frac{2}{\pi}x + 1\right) dx$ = AREA WE WANT

$$A = \sin x \Big|_{-\frac{\pi}{2}}^{0} - \left(\frac{2}{\pi}\frac{1}{2}x^2 + x\right)\Big|_{-\frac{\pi}{2}}^{0}$$

$$A = \left(\cancelto{0}{\sin 0} - \cancelto{-1}{\sin\left(-\frac{\pi}{2}\right)}\right) - \left[\left(\cancelto{0}{\frac{1}{\pi}(0)^2} + \cancelto{0}{(0)}\right) - \left(\frac{1}{\pi}\left(\frac{-\pi}{2}\right)^2 + \left(\frac{-\pi}{2}\right)\right)\right]$$

$$A = 1 - \left(\frac{\pi}{4} - \frac{\pi}{2}\right) = 1 - \left(\frac{\pi}{4} - \frac{2\pi}{4}\right)$$

$$= 1 - \left(-\frac{\pi}{4}\right)$$

$$\boxed{A = 1 + \frac{\pi}{4}}$$

8) EVALUATE DEFINITE INTEGRALS:

a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{4}{\sqrt{1-x^2}}\,dx = 4\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= 4\left(\sin^{-1}x\Big|_{\frac{\sqrt{2}}{2}}^{\sqrt{3}/2}\right) = 4\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$$

$$= \boxed{4\left(\frac{\pi}{3} - \frac{\pi}{4}\right)}$$

b) $$\int_1^2 \frac{2}{3x+1}\,dx = 2\int_1^2 \frac{1}{3x+1}\,dx$$

let $u = 3x+1$

$du = 3\,dx$

$dx = \frac{du}{3}$

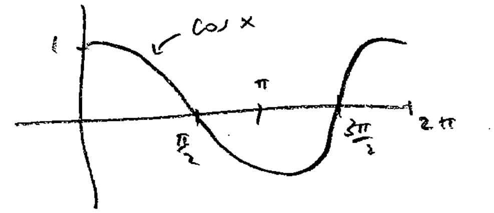
CHANGE BOUNDS:

$$\left\{ \begin{array}{l} x = 1 \rightarrow u = 4 \\ x = 2 \rightarrow u = 7 \end{array} \right\}$$

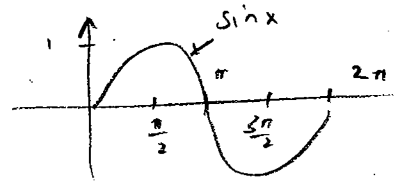
$$\Rightarrow \quad = 2\int_4^7 \frac{1}{u}\,\frac{du}{3} = \frac{2}{3}\int_4^7 \frac{1}{u}\,du = \frac{2}{3}\left(\ln|u|\Big|_4^7\right)$$

$$= \frac{2}{3}\left(\ln 7 - \ln 4\right) = \boxed{\frac{2}{3}\ln\frac{7}{4}}$$

c) $\int_0^{\pi} (\sin x + \cos x)\, dx$



$= \int_0^{\pi} \sin x\, dx + \int_0^{\pi} \cos x\, dx$



$= -\cos x \Big|_0^{\pi} + \sin x \Big|_0^{\pi}$

$= (1 + 1) + (0 - 0) = \boxed{2}$

9 a) SEPERABLE DIFFERENTIAL EQN.
OR INITIAL VALUE PROBLEM

$\frac{dL}{dt} = \sqrt{2t+4}$

$dL = \sqrt{2t+4}\, dt$ // GET ALL "L's" ON ONE SIDE AND ALL "t" TERMS ON OTHER

$\int dL = \int \sqrt{2t+4}\, dt$ / INTEGRATE BOTH SIDES

$L = \int \sqrt{2t+4}\, dt$ let $u = 2t+4$

$du = 2dx$

$dx = \frac{du}{2}$

$L = \frac{1}{2} \int u^{1/2}\, du = \frac{1}{2} \frac{2}{3} u^{3/2} + c$

$L(t) = \frac{1}{3} (2t+4)^{3/2} + C$ / NOW FIND C W/ INITIAL VALUE

GIVEN : $L(0) = 98$

PLUG IT IN!

$$L(0) = 98 = \frac{1}{3}(\cancel{2}(0)^{0} + 4)^{3/2} + C$$

$$98 = \frac{1}{3} 8 + C$$

$$C = 98 - \frac{1}{3} 8 = \frac{294}{3} - \frac{8}{3} = \frac{286}{3}$$

$$\boxed{L(t) = \frac{1}{3}(2t + 4)^{3/2} + \frac{286}{3}}$$

b) AVERAGE VALUE $= \frac{1}{b-a} \int_a^b f(x)\,dx$

$$AVG(L(t)) = \frac{1}{16-6}\left[\int_6^{16} \left(\frac{1}{3}(2t+4)^{3/2} + \frac{286}{3}\right) dx\right]$$

$$= \frac{1}{10}\left[\frac{1}{3}\int_6^{16} (2t+4)^{3/2}\,dx + \frac{286}{3}\int_6^{16} dx\right]$$

$\Rightarrow$ let $u = 2t+4$ $\quad t = 6 \quad u = 32$

$du = 2dt \quad t = 16 \quad u = 216$

$dt = \frac{du}{2}$

$$= \frac{1}{10}\left[\frac{1}{6}\int_{32}^{216} u^{3/2}\,du + \frac{286}{3} t \Big|_6^{16}\right]$$

... PLUG IN #'s