Math 11B
Calculus with Applications
Fall 2007
Midterm Exam 2

Solutions

1. (20 Points) Evaluate the following indefinite integrals

   a. (10 Points) $\int \frac{1}{x^2-4}\,dx$

   Solution:

   $$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} \quad \Rightarrow \quad 1 = A(x+2) + B(x-2) = (A+B)x + (2A-2B)$$

   $$\Rightarrow \begin{cases} A + \ \ B = 0 \\ 2A - 2B = 1 \end{cases} \Rightarrow \ A = \frac{1}{4},\ B = -\frac{1}{4}$$

   $$\Rightarrow \int \frac{1}{x^2-4}\,dx \ = \ \int \left(\frac{1/4}{x-2} - \frac{1/4}{x+2} \right) dx \ = \ \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C \ = \ \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

   b. (10 Points) $\int t^2 e^t\,dt$

   Solution:

   $$\begin{cases} u = t^2 & dv = e^t\,dt \\ du = 2t\,dt & v = e^t \end{cases} \quad \Rightarrow \quad \int t^2 e^t\,dt = t^2 e^t - 2\int t e^t\,dt$$

   $$\begin{cases} u = t & dv = e^t\,dt \\ du = dt & v = e^t \end{cases} \quad \Rightarrow \quad \int t^2 e^t\,dt = t^2 e^t - 2\left(te^t - \int e^t\,dt \right) = \ t^2 e^t - 2te^t + 2e^t + C = e^t(t^2 - 2t + 2) + C$$

2. (20 Points) Evaluate the following definite integrals

   a. (10 Points) $\int_{\pi^2/16}^{\pi^2/9} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

   Solution:

   $$\begin{cases} u = \sqrt{x} & x = \pi^2/9 \ \Rightarrow u = \pi/3 \\ du = \frac{1}{2\sqrt{x}}dx & x = \pi^2/16 \ \Rightarrow u = \pi/4 \\ 2du = \frac{1}{\sqrt{x}}dx & \end{cases}$$

   $$\int_{\pi^2/16}^{\pi^2/9} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx \ = \ 2\int_{\pi/4}^{\pi/3} \cos u\,du \ = \ 2\sin u \Big|_{\pi/4}^{\pi/3} \ = \ 2(\sin(\pi/3) - \sin(\pi/4)) = 2\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \right) = \sqrt{3} - \sqrt{2}$$

b. (10 Points) $\int_1^2 \frac{x^3}{x^2+1}\,dx$

Solution:

$$\begin{array}{r} x \\ x^2+0x+1\overline{\big)\,x^3+0x^2+0x+0} \\ \underline{x^3+0x^2+x} \\ -x \end{array} \qquad \Rightarrow \quad \frac{x^3}{x^2+1} = x + \frac{-x}{x^2+1} = x - \frac{1}{2}\left(\frac{2x}{x^2+1}\right)$$

$$\int_1^2 \frac{x^3}{x^2+1}\,dx = \int_1^2 \left(x - \frac{1}{2}\cdot\frac{2x}{x^2+1}\right)dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1)\,\Big|_1^2 = \frac{1}{2}\big((4-\ln 5)-(1-\ln 2)\big) = \frac{1}{2}\left(3+\ln\left(\frac{2}{5}\right)\right)$$

3. (20 Points) Solve the following initial value problem $\frac{dH}{dt} = \frac{\ln t}{t}$ for $t > 1$, and $H(e) = 0$.

Solution:

$$\begin{cases} u = \ln t \\ du = \frac{1}{t}dt \end{cases} \Rightarrow \quad H(t) = \int \frac{\ln t}{t}dt = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln t)^2 + C$$

$$0 = H(e) = \frac{1}{2} + C \quad \Rightarrow \quad C = -\frac{1}{2} \quad \Rightarrow \quad H(t) = \frac{1}{2}(\ln t)^2 - \frac{1}{2}$$

4. (20 Points) Determine whether the following improper integrals converge or diverge. If the integral converges, determine it's value.

a. (10 Points) $\int_0^1 \frac{1}{x^2}\,dx$

Solution:

The improper integral diverges since:

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{c\to 0^+}\int_c^1 x^{-2}\,dx = \lim_{c\to 0^+}(-x^{-1})\,\Big|_c^1 = -\lim_{c\to 0^+}\left(1-\frac{1}{c}\right) = \infty,$$

b. (10 Points) $\int_0^1 (x-1)^{-1/3}\,dx$

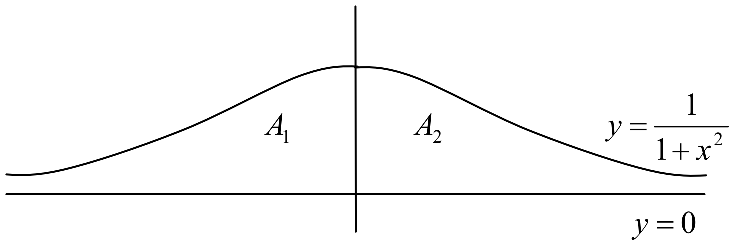
Solution:

The improper integral converges to $-\frac{3}{2}$ since:

$$\int_0^1 (x-1)^{-1/3}\,dx = \lim_{c\to 1^-}\int_0^c (x-1)^{-1/3}\,dx = \lim_{c\to 1^-}\frac{3}{2}(x-1)^{2/3}\,\Big|_0^c = \frac{3}{2}\lim_{c\to 1^-}\left((c-1)^{2/3}-(-1)^{2/3}\right) = \frac{3}{2}(-1) = -\frac{3}{2}$$

5. (20 Points) Determine the area bounded by the curves $y = \frac{1}{1+x^2}$, and $y = 0$.

Solution:



By symmetry we see that $A_1 = A_2$, whence

$$A = A_1 + A_2 = 2A_2 = 2\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{z\to\infty} 2\int_0^z \frac{1}{1+x^2}\,dx = 2\lim_{z\to\infty} \tan^{-1}(x)\Big|_0^z$$

$$= 2\lim_{z\to\infty}\left(\tan^{-1}(z) - \tan^{-1}(0)\right) = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

6. (20 Points) Determine the indefinite integral $\int (\ln x)^2\,dx$ by first doing the substitution $t = \ln x$, then referring to problem 1b.

Solution:

$$\begin{cases} t = \ln x \;\rightarrow\; x = e^t \\ dt = \frac{1}{x}dx \;\rightarrow\; x\,dt = dx \;\rightarrow\; e^t\,dt = dx \end{cases}$$

$$\Rightarrow \int (\ln x)^2\,dx = \int t^2 e^t\,dt = e^t(t^2 - 2t + 2) + C = x\left((\ln x)^2 - 2(\ln x) + 2\right) + C$$

$\uparrow$ by problem 1b

Some Indefinite Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{(for } n \neq -1\text{)}$$

$$\int x^{-1} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \sec^2 x\, dx = \tan x + C$$

$$\int \csc^2 x\, dx = -\cot x + C$$

$$\int \sec x \tan x\, dx = \sec x + C$$

$$\int \csc x \cot x\, dx = -\csc x + C$$

$$\int \tan x\, dx = \ln|\sec x| + C$$

$$\int \cot x\, dx = -\ln|\csc x| + C$$

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$$

Standard Angles